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Holography in superspace

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Abstract: The AdS/CFT correspondence identifies the coordinates of the conformal boundary of anti-de Sitter space with the coordinates of the conformal field theory. We generalize this identification to theories formulated in superspace. As an application of our results, we study a class of Wilson loops in $\mathcal{N} = 4$ super-Yang-Mills theory. A gauge theory computation shows that the expectation values of these loops are invariant under a local κ-symmetry, except at intersections. We identify this with the κ-invariance of the associated string worldsheets in the corresponding bulk superspace.

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1. Introduction

One of the key ingredients in the AdS/CFT correspondence [1] is the identification of the conformal boundary of anti-de Sitter space with the space on which the conformal field theory is defined. In AdS space with Poincaré coordinates (x^μ, y) and metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2)\,, \tag{1.1}$$

the boundary is located at $y = 0$, and we identify the x^μ as the coordinates of the dual conformal field theory. Given this identification of coordinates, the isometries of AdS, restricted to the boundary, are identified with the conformal symmetries of the boundary. Consider, for example, the infinitesimal isometry

$$\delta x^\mu = -2(\epsilon \cdot x)x^\mu + \epsilon^\mu (x^2 + y^2)\,, \tag{1.2}$$

$$\delta y = -2(\epsilon \cdot x) y\,, \tag{1.3}$$

which preserves the metric (1.1). As $y \to 0$, the coordinate y decouples from the isometry action, and δx^μ reduces to a special conformal transformation of the boundary.

Do similar relations hold in bulk-boundary pairs of superspaces? One would expect any such relations to involve the fermionic coordinates in an interesting way. (For example, $AdS_5 \times S^5$ superspace has 32 fermionic coordinates, but there are only 16 fermionic coordinates in the boundary theory.) Bosonic AdS space and its conformal boundary can be realized as coset manifolds of the same group. This suggests studying the bulk-boundary correspondence in superspace from the point of view of coset supermanifolds. In section 2, we show that, under certain conditions, the symmetries of a bulk-boundary pair of coset supermanifolds coincide in an appropriately defined boundary limit. We then delve into the identification of coordinates and symmetries in the case that the bulk supermanifold is the (10|32)-dimensional $AdS_5 \times S^5$ superspace whose bosonic part is $AdS_5 \times S^5$, and the boundary is the conformal superspace on which the $\mathcal{N} = 4$ super-Yang-Mills theory is naturally defined.[1]

We next apply the superspace approach to the correspondence between Wilson loops in $\mathcal{N} = 4$ super-Yang-Mills theory and string worldsheets in $AdS_5 \times S^5$ [3, 4]. In section 3, we define the Wilson loop operator W in the superfield formulation of $\mathcal{N} = 4$ super-Yang-Mills theory. If the loop is lightlike (that is, if its tangent vector in superspace is everywhere null), W is invariant under a local κ-symmetry, provided the classical super-Yang-Mills equations of motion are satisfied. The κ-symmetry in question is defined in section 3 and is essentially the usual κ-symmetry of a massless superparticle. Of course, the quantity that enters into the AdS/CFT correspondence is not the classical operator W, but rather its quantum expectation value $\langle W \rangle$. Replacing classical equations of motion by Schwinger-Dyson equations, we are able to show that $\langle W \rangle$, to lowest nontrivial order in an expansion in fermionic superspace coordinates, is κ-invariant, provided the loop is lightlike and smooth and contains no self-intersections. On the other hand, we find that the κ-symmetry is violated when the loop has intersections.

According to the AdS/CFT correspondence, Wilson loops in the boundary gauge theory are associated with worldsheets in the bulk string theory. We employ the approach of [5, 6] to study the Green-Schwarz superstring propagating in the $AdS_5 \times$

[1] Our analysis continues the study of the symmetries of these spaces begun in [2].

S^5 superspace.[2] The bulk string worldsheets are constrained to end on the Wilson loop, which lives in the conformal boundary of the AdS space. In other words, the Wilson loop imposes boundary conditions on the string worldsheet with which it is associated [3, 4]. Using the coordinate identifications worked out in section 2, we study these boundary conditions in section 4, extending to superspace the analysis carried out in [10] for bosonic loops. We learn, for example, that the requirement that the loop be lightlike, which we arrived at in section 3 for wholly gauge-theoretic reasons, has an interpretation in the context of AdS/CFT as a necessary condition for the string worldsheet to terminate on the boundary of the AdS space. Finally, we show that the κ-symmetry variation of the Wilson loop in the gauge theory can be understood as the restriction to the boundary of the κ-symmetry of the associated string worldsheet in $AdS_5 \times S^5$ superspace.

1.1 Notation

Indices.

$$\begin{aligned}
\hat{\mu}, \hat{\nu} &= 0, \ldots, 9 \text{ are vector indices in 10 dimensions.} \\
\mu, \nu &= 0, \ldots, 3 \text{ are vector indices in 4 dimensions.} \\
m, n &= 4, \ldots, 9 \text{ are vector indices in 6 dimensions.}
\end{aligned}$$

$$\begin{aligned}
\hat{\alpha}, \hat{\beta} &= 1, \ldots, 16 \text{ are Weyl spinor indices in 10 dimensions.} \\
\alpha, \beta &= 1, \ldots, 4 \text{ are Dirac spinor indices in 4 dimensions.} \\
a, b &= 1, \ldots, 4 \text{ are Weyl spinor indices in 6 dimensions.}
\end{aligned}$$

These are indices for general coordinates. The indices for the local Lorentz frame are obtained by underlining the corresponding general coordinate indices, e.g., $\mu \to \underline{\mu}$.

Coordinates. The (10|16)-dimensional *boundary* superspace has coordinates

$$z^M = (x^{\hat{\mu}}, \lambda^{\hat{\alpha}}) = (x^\mu, y^m, \lambda^{\hat{\alpha}}) \,. \tag{1.4}$$

The superspace derivatives are

$$\begin{aligned}
\partial_{\hat{\mu}} &= \frac{\partial}{\partial x^{\hat{\mu}}} \,, \\
D_{\hat{\alpha}} &= \frac{\partial}{\partial \lambda^{\hat{\alpha}}} + (\Gamma^{\hat{\mu}}\lambda)^{\hat{\alpha}} \frac{\partial}{\partial x^{\hat{\mu}}} \,, \\
Q_{\hat{\alpha}} &= \frac{\partial}{\partial \lambda^{\hat{\alpha}}} - (\Gamma^{\hat{\mu}}\lambda)^{\hat{\alpha}} \frac{\partial}{\partial x^{\hat{\mu}}} \,, \\
\mathbf{D} &= \lambda^{\hat{\alpha}} \frac{\partial}{\partial \lambda^{\hat{\alpha}}} = \lambda^{\hat{\alpha}} D_{\hat{\alpha}} \,.
\end{aligned} \tag{1.5}$$

[2]Because of the κ-symmetry, the approach of [6] is difficult to use to describe short strings in the bulk, but has been useful in studying string worldsheets of macroscopic size ending on Wilson loops on the boundary [7, 8, 9].

The (10|32)-dimensional *bulk* superspace has coordinates

$$Z^{\mathbf{M}} = (X^{\hat{\mu}}, \theta^{\hat{\alpha}}, \vartheta^{\hat{\alpha}}) = (X^{\mu}, Y^{m}, \theta^{\hat{\alpha}}, \vartheta^{\hat{\alpha}})\,. \tag{1.6}$$

Sometimes we write $Y^m = (Y, \phi^m)$, where Y is the radial coordinate in the AdS space and the ϕ^m's are coordinates on the 5-sphere.

2. Supercosets and holography

It is well known that the isometries of AdS reduce on the boundary to conformal transformations. As we saw in the example (1.3), the radial coordinate y decouples from the transformations of the coordinates parallel to the boundary. These parallel coordinates are naturally identified with the coordinates of the boundary conformal field theory. Adding supersymmetry makes the boundary limit analysis more interesting. Consider, for example, (10|32)-dimensional $AdS_5 \times S^5$ superspace. In addition to y, half of the fermionic variables must decouple from the isometries at the boundary: the bulk theory has 32 supersymmetries, whereas the boundary theory, being a non-gravitational theory, admits only 16 linear supersymmetries. Thus, the 32 fermions of the bulk theory split into two sets of 16. One set decouples from the isometries in the boundary limit, and the other set maps in this limit to the 16 fermionic coordinates of the conformal field theory.

In this section, we analyze the identification of coordinates and symmetries from the perspective of coset manifolds. This point of view is fruitful for the generalization to superspace, which is our main interest. In section 2.1, we briefly review coset techniques. In section 2.2, we exhibit the boundary reduction of AdS_5 isometries to four-dimensional conformal symmetries, presenting the ideas from the coset manifold point of view. Section 2.3 develops a general formalism for comparing the symmetries of two coset manifolds of the same group. Finally, in section 2.4, we apply our general techniques to the (10|32)- and (10|16)-dimensional supercoset manifolds that enter in the superspace formulation of the AdS_5/CFT_4 correspondence. We match coordinates and symmetries at the boundary, verify that half of the 32 AdS fermionic coordinates decouple, and discuss the relation of the supervielbeins of the two spaces.

2.1 A brief review of supercosets

Let $C = G/H$ be a coset (or supercoset) manifold with coordinates Z^M, and let us choose the coset representative $c(Z) \in G$. The coset representative is defined up to the equivalence relation

$$c(Z) \sim c(Z)h(Z)\,, \tag{2.1}$$

where $h(Z)$ is a local H transformation. The Lie algebra-valued Cartan 1-form $\mathcal{L}$ is defined as

$$\mathcal{L}(Z) = \mathcal{L}^A(Z)\mathbf{g}_A = dZ^M \mathcal{L}_M{}^A \mathbf{g}_A \equiv c(Z)^{-1} dc(Z)\,, \tag{2.2}$$

where the $\mathbf{g}_A$ are the generators of G. The Cartan form $\mathcal{L}$ can be written as

$$\mathcal{L}^A \mathbf{g}_A = E + \omega = E^{\underline{M}} \mathbf{c}_{\underline{M}} + \omega^i \mathbf{h}_i \,. \tag{2.3}$$

In this expression, $\mathbf{c}_{\underline{M}}$ and $\mathbf{h}_i$ denote the coset and stability group generators, respectively; their coefficients $E^{\underline{M}}$ and ω^i are the supervielbein and the $\mathbf{h}$-connection, which is a generalization of the usual spin connection. The equivalence relation (2.1) induces the identifications

$$\begin{aligned} E^{\underline{M}} &\sim \big(h(Z)^{-1} E h(Z)\big)^{\underline{M}} \,, \\ \omega^i &\sim \big(h(Z)^{-1} \mathcal{L} h(Z)\big)^i + h(Z)^{-1} dh(Z) \,. \end{aligned} \tag{2.4}$$

In particular, the vielbein is defined only up to local H-transformations.

Under left-multiplication by a constant $g \in G$, the coset representative transforms as

$$c(Z) \to c(Z') = g c(Z) h(Z)^{-1} \,. \tag{2.5}$$

The compensating transformation $h(Z)$ ensures that $c(Z')$ remains in the same gauge slice. The infinitesimal form of (2.5) reads

$$\delta c(Z) = \mathbf{g} c(Z) - c(Z) \mathbf{h}(Z) \,, \tag{2.6}$$

where

$$\begin{aligned} g &= 1 + \mathbf{g} \,, \\ h &= 1 + \mathbf{h}(Z) \,. \end{aligned} \tag{2.7}$$

The left-multiplication (2.5) induces a transformation of the coordinates Z^M via

$$\delta c(Z) = \delta Z^M \partial_M c(Z) \,. \tag{2.8}$$

These global symmetries leave the vielbein invariant up to the local $\mathbf{h}$-transformation

$$E^{\underline{M}} \to E^{\underline{M}} + [\mathbf{h}(Z), E]^{\underline{M}} . \tag{2.9}$$

If G and H are semisimple bosonic groups, the unique and natural line element is invariant under symmetries of this form. In this case, these symmetries are isometries in the usual sense.

2.2 AdS/CFT: the bosonic case

Now let us turn to the bosonic cosets involved in the AdS_5/CFT_4 correspondence. The bulk space AdS_5 is the coset manifold $G/H = \mathrm{SO}(2,4)/\,\mathrm{SO}(1,4)$, and is parametrized by four boundary coordinates x^μ and the radial coordinate y. The symmetries of the boundary $\mathbb{R}^4$ are made manifest if we realize this space as the coset $G/H' =$

Operator	Weight	Name
P_μ	1	Conformal Translations
$M_{\mu\nu}$	0	Lorentz Rotations
D	0	Dilatation
K_μ	-1	Special Conformal Transformations

Table 1: SO$(2,4)$ generators in the conformal basis.

SO$(2,4)/\,$Span$(\mathrm{iso}(1,3)_K \oplus D)$, which we refer to as conformal space. Here Span$(\cdots)$ denotes the group generated by the operators in $(\cdots)$. The stability group H' is not semi-simple, and the G-action preserves the natural metric on $\mathbb{R}^4$ only up to rescaling. The SO$(2,4)$ generators and their conformal weights (with respect to the dilatation operator D) are listed in table 1. The classification of generators by coset and stability group is given in table 2.

	AdS_5	$\mathbb{R}^4$
C	$\frac{1}{2}(P_\mu + K_\mu), \quad D$	P_μ
H	$\frac{1}{2}(P_\mu - K_\mu), \quad M_{\mu\nu}$	$K_\mu, \quad M_{\mu\nu}$

Table 2: Coset decompositions of SO$(2,4)$ Generators.

To relate AdS_5 and conformal space, we first select coset representatives. A convenient choice of representative for the AdS coset is given [13] by

$$c_{AdS}(x) = e^{x^\mu P_\mu} y^D \,. \tag{2.10}$$

The advantage of this choice is that the coset generators are ordered by weight, which simplifies the form of the Cartan form and the symmetry transformations. For the coset representative of the conformal space $\mathbb{R}^4$ we choose the standard form

$$c_{cf}(x) = e^{x^\mu P_\mu} \,. \tag{2.11}$$

Strictly speaking, we should introduce different symbols $\tilde{x}$ to denote coordinates for this space, but as we show in the next subsection, both the vielbeins in the x directions and the symmetries agree in the limit $y \to 0$, so we allow this imprecise notation for better readability.

The AdS coset representative (2.10) gives rise to the AdS vielbein [13]

$$E^{\underline{\mu}} = \frac{dx^{\underline{\mu}}}{y}, \qquad E^{\underline{y}} = \frac{dy}{y} \,. \tag{2.12}$$

The vielbein e of conformal space is

$$e^{\underline{\mu}} = dx^{\underline{\mu}}, \tag{2.13}$$

which agrees with the parallel components of (2.12) up to a conformal rescaling. This conformal rescaling is a “gauge transformation” of precisely the form (2.9), with $\mathbf{h} = \frac{1-y}{y} D$.

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Let us revisit the well-known correspondence between bulk isometries and boundary conformal symmetries from the point of view of the coset construction. Consider the special conformal transformation with $\mathbf{g} = \epsilon^\mu K_\mu$. The action of this symmetry on the coordinates of conformal space is

$$\delta_{cf} x^\mu = x^2 \epsilon^\mu - 2(\epsilon \cdot x) x^\mu \,. \tag{2.14}$$

To calculate the corresponding isometry of the AdS coset, we note that the gauge choice (2.10) requires that the $\mathbf{g}$-action be accompanied by a compensating $\mathbf{h}$-transformation of the form

$$\mathbf{h}_{AdS} = -4 x^\mu \epsilon^\nu M_{\mu\nu} - y \epsilon^\mu (P_\mu - K_\mu) \,. \tag{2.15}$$

We see that $\mathbf{h}_{AdS}$ includes a modifying translation, because the stability group includes the combination $\frac{1}{2}(P_\mu - K_\mu)$. From here, we can read off

$$\delta_{AdS} x^\mu = x^2 \epsilon^\mu - 2(\epsilon \cdot x) x^\mu + y^2 \epsilon^\mu \,. \tag{2.16}$$

Therefore,

$$\Delta(\delta x^\mu) \equiv \delta_{AdS} x^\mu - \delta_{conf} x^\mu = y^2 \epsilon^\mu \,, \tag{2.17}$$

which vanishes on the boundary $y \to 0$. It is worthwhile to note that the crucial condition for the agreement between the two variations is

$$\left(y^D \mathbf{h}_{AdS}(x,y) y^{-D} \right)\big|_{P_\mu} \sim y^2 \to 0 \,. \tag{2.18}$$

2.3 The general picture

We now generalize the above discussion to arbitrary supercosets. Consider two cosets $C_1 = G/H_1$ and $C_2 = G/H_2$ with the same underlying group G, but different stability groups H_1 and H_2. Let their coordinates be by $Z^M = (x^m, y^i)$ and x^m, respectively (again, this notation anticipates that the x^m coordinates of both spaces can be identified in a suitable limit). In this subsection only, x and y can denote either bosonic or fermionic coordinates. We further suppose that the coset representative of C_1 has the form

$$c_1(x,y) = c_2(x) h_2(x,y) \,, \tag{2.19}$$

where $h_2(x,y)$ is an (x,y)-dependent element of H_2 and $c_2(x)$ is a coset representative of C_2.

How are the symmetries of C_1 and C_2 related? To begin with,

$$c^{-1}(\mathbf{g}\, c - c\mathbf{h}) = \delta z^M \mathcal{L}_M \,, \tag{2.20}$$

where

$$\mathcal{L}_M = E_M{}^{\underline{m}} \mathbf{c}_{\underline{m}} + \omega_M{}^i \mathbf{h}_i \,. \tag{2.21}$$

Applying (2.20) to the coset C_1 and making use of the factorization (2.19) gives

$$c_2^{-1}(\mathbf{g}c_1 - c_1\mathbf{h}_1)h_2^{-1} = \delta_1 x^m(\mathcal{L}_m^{(2)} + (\partial_m h_2)h_2^{-1}) + \delta_1 y^i(\partial_i h_2)h_2^{-1}\,, \tag{2.22}$$

where $\mathbf{h}_1 = \mathbf{h}_1(\mathbf{g}, x, y)$ is the compensating transformation for G within coset 1, the expressions $\mathcal{L}_m^{(2)} = c_2^{-1}(x)\partial_m c_2(x)$ are components of the Cartan form of C_2, and the notation δ_1 reminds us that we are considering symmetries of C_1. On the other hand,

$$\begin{aligned} c_2^{-1}(\mathbf{g}c_1 - c_1\mathbf{h}_1)h_2^{-1} &= c_2^{-1}\mathbf{g}c_2 - h_2\mathbf{h}_1 h_2^{-1} \\ &= c_2^{-1}\mathbf{g}c_2 - \mathbf{h}_2 + \mathbf{h}_2 - h_2\mathbf{h}_1 h_2^{-1} \\ &= \delta_2 x^m \mathcal{L}_m^{(2)} + \mathbf{h}_2 - h_2\mathbf{h}_1 h_2^{-1}\,. \end{aligned} \tag{2.23}$$

Here $\mathbf{h}_2 = \mathbf{h}_2(\mathbf{g}, x, y)$ is the compensating transformation for $\mathbf{g}$ in coset 2, and $\delta_2 x^m$ is the corresponding infinitesimal coordinate variation. From (2.22) and (2.23), it follows that

$$\begin{aligned} \Delta(\delta x^m)\mathcal{L}_m^{(2)} &\equiv (\delta_1 x^m - \delta_2 x^m)\mathcal{L}_m^{(2)} \\ &= \mathbf{h}_2 - h_2\mathbf{h}_1 h_2^{-1} - \delta_1 x^m(\partial_m h_2)h_2^{-1} - \delta_1 y^i(\partial_i h_2)h_2^{-1}\,. \end{aligned} \tag{2.24}$$

We now compare the coefficients of the C_2 generators on both sides of (2.24). On the right-hand side, the only term with a potentially nonzero coefficient is $-h_2\mathbf{h}_1 h_2^{-1}$; the other terms lie exclusively in the Lie algebra of the stability group H_2. Consequently,

$$\Delta(\delta x^m)E_m^{(2)\underline{m}} = -\left(h_2\mathbf{h}_1 h_2^{-1}\right)\Big|_{C_{\underline{m}}^{(2)}}\,. \tag{2.25}$$

The vielbein $E^{(2)}$ is invertible and is independent of y. The symmetries of C_1 and C_2 agree, i.e., $\Delta(\delta x^m) = 0$, if

$$\left(h_2\mathbf{h}_1 h_2^{-1}\right)\Big|_{C_{\underline{m}}^{(2)}} = 0\,, \tag{2.26}$$

at some value of y ($y = 0$, in the AdS example). In this case, we say C_2 is the boundary limit of C_1, located at $y = 0$. This condition generalizes (2.18).

2.4 AdS/CFT: the supersymmetric case

Both $AdS_5 \times S^5$ superspace and conformal superspace are supercosets of $G = \mathrm{SU}(2, 2|4)$, but with different stability groups. Table 3 lists the generators of $\mathrm{SU}(2,2|4)$ with their weights under the dilatation operator D.

The $AdS_5 \times S^5$ supercoset

$$C_{(10|32)} = \frac{\mathrm{SU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)} \tag{2.27}$$

is (10|32)-dimensional, as the stability group contains no fermionic generators. On the other hand, the conformal superspace

$$C_{(10|16)} = \frac{\mathrm{SU}(2,2|4)}{\mathrm{Span}\,(\mathrm{iso}(1,3)_K \oplus \mathrm{so}(5) \oplus S)}\,, \tag{2.28}$$

Operator	Weight	Name
P_μ	1	Conformal Translations
Q	1/2	Global Supersymmetries
$M_{\mu\nu}$	0	Lorentz Rotations
D	0	Dilatation
$U_j{}^i = (\widetilde{M}_{m'n'}, \widetilde{P}_{m'})$	0	SU(4) Rotations of S^5
S	$-1/2$	Special Supersymmetries
K_μ	-1	Special Conformal Transformations

Table 3: SU(2, 2|4) generators in the superconformal basis. The generators of SU(4) rotations may be written as $U_i^j = 2\widetilde{P}_{m'}(\widetilde{\Gamma}^{m'6})_i^j + \widetilde{M}_{m'n'}(\widetilde{\Gamma}^{m'n'})_i^j$, where the $\widetilde{P}_{m'}$ and $\widetilde{M}_{m'n'}$ $(m', n' = 1, \ldots, 5)$ are generators of translations and rotations on S^5, and the $\widetilde{\Gamma}$'s are the 4×4 chiral blocks of the SO(6) Dirac matrices in the chiral basis.

	$C_{(10\|32)}$	$C_{(10\|16)}$	$C_{(4\|16)}$
C	$\frac{1}{2}(P_\mu + K_\mu), \widetilde{P}_{m'}, D, Q, S$	$P_\mu, \quad \widetilde{P}_{m'}, \quad D, \quad Q$	$P_\mu, \qquad Q$
H	$\frac{1}{2}(P_\mu - K_\mu), M_{\mu\nu}, \widetilde{M}_{m'n'}$	$K_\mu, M_{\mu\nu}, \widetilde{M}_{m'n'}, S$	$K_\mu, M_{\mu\nu}, \widetilde{P}_{m'}, \widetilde{M}_{m'n'}, D, S$

Table 4: Coset decompositions of SU(2, 2|4) Generators.

is (10|16)-dimensional. This space is an extension of the (4|16)-dimensional superspace

$$C_{(4|16)} = \frac{\mathrm{SU}(2,2|4)}{\mathrm{Span}\,(\mathrm{iso}(1,3)_K \oplus D \oplus \mathrm{so}(6) \oplus S)}\,. \tag{2.29}$$

In table 4 we give the division of the SU(2, 2|4) generators into coset and stability group generators for each coset space.

2.4.1 The supergeometry at the boundary

A representative of the $AdS_5 \times S^5$ supercoset that is convenient for our purpose is given by

$$\begin{aligned} c_{(10|32)}(x, Y, \phi, \theta, \vartheta) &= e^{x^\mu P_\mu} e^{\bar{Q}\theta + \bar{\theta}Q} e^{\bar{S}\vartheta + \bar{\vartheta}S} u(\phi) Y^D \\ &= e^{x^\mu P_\mu} e^{\bar{Q}\theta + \bar{\theta}Q} u(\phi) Y^D \times Y^{-D} u(\phi)^{-1} e^{\bar{S}\vartheta + \bar{\vartheta}S} u(\phi) Y^D \\ &= e^{x^\mu P_\mu} e^{\left(\bar{Q}_a^\alpha \theta_\alpha^a + \bar{\theta}_a^\alpha Q_\alpha^a\right)} u(\phi) Y^D \times e^{\sqrt{Y}\left(\bar{S}_a^\alpha \vartheta_\alpha^b u_b{}^a + (u^{-1})_a{}^b \bar{\vartheta}_b^\alpha S_\alpha^a\right)}\,. \end{aligned} \tag{2.30}$$

The matrices $u_a{}^b(\phi)$ are coset representatives of the SO(6)/ SO(5) subcoset. The first factor in the last line of (2.30) is the coset representative of the (10|16)-dimensional conformal superspace,

$$c_{(10|16)}(x, Y, \phi, \theta) = e^{x^\mu P_\mu} e^{\left(\bar{Q}_a^\alpha \theta_\alpha^a + \bar{\theta}_a^\alpha Q_\alpha^a\right)} u(\phi) Y^D\,, \tag{2.31}$$

and the second factor is in the stability group of $\mathcal{C}_{(10|16)}$, since it contains generators of negative weights only. Hence (2.30) is of the form (2.19),

$$c_{(10|32)} = c_{(10|16)}(x, Y, \phi, \theta) \times h_{(10|16)}(x, \theta, \vartheta, Y, \phi) \,. \tag{2.32}$$

By construction, then, the supervielbeins of $C_{(10|16)}$ are related to those of $C_{(10|32)}$ by a coordinate-dependent gauge transformation, as in (2.4). We now calculate these supervielbeins and exhibit this gauge transformation explicitly. We will use the results in section 4.

The Cartan form $\mathcal{L}^{(10|16)}$ of conformal superspace decomposes as

$$\mathcal{L}^{(10|16)} = c^{-1}_{(10|16)} dc_{(10|16)} = e^{\underline{\mu}} P_{\underline{\mu}} + e^Y D + e^{m'} \widetilde{P}_{m'} + \bar{e}^\alpha_a Q^a_\alpha + \bar{Q}^\alpha_a e^a_\alpha + \omega^i \mathbf{h}_i \,, \tag{2.33}$$

where ω^i is the $\mathbf{h}$-connection. The supervielbein is determined to be

$$\begin{aligned}
e^{\underline{\mu}} &= \frac{1}{Y}\left[dx^{\underline{\mu}} + \frac{1}{2}\left(d\bar{\theta}\gamma^{\underline{\mu}}\theta - \bar{\theta}\gamma^{\underline{\mu}}d\theta\right)\right],\\
e^{\underline{Y}} &= \frac{dY}{Y}\,,\\
e^{\underline{m}} &= e^{\underline{m}'}(\phi),\\
e^{\underline{a}}_{\underline{\alpha}} &= Y^{-1/2} d\theta^{\underline{b}}_{\underline{\alpha}} u(\phi)_{\underline{b}}{}^{\underline{a}}\,,\\
\bar{e}^{\underline{\alpha}}_{\underline{a}} &= Y^{-1/2} u^{-1}(\phi)_{\underline{a}}{}^{\underline{b}} d\bar{\theta}^{\underline{\alpha}}_{\underline{b}}\,.
\end{aligned} \tag{2.34}$$

The connection has components only in the sphere directions,

$$\omega = \omega^{m'n'}(\phi)\widetilde{M}_{m'n'} \,. \tag{2.35}$$

Given the coset representative (2.30), it is straightforward to calculate the Cartan form $\mathcal{L}^{(10|32)}$ of the $AdS_5 \times S^5$ superspace. Since the coset representative c is of the form (2.30), the relation between the $\mathcal{L}^{(10|32)}$ and $\mathcal{L}^{(10|16)}$ is

$$\mathcal{L}^{(10|32)} = h^{-1}_{(10|16)} \mathcal{L}^{(10|16)} h_{(10|16)} + h^{-1}_{(10|16)} dh_{(10|16)} \,. \tag{2.36}$$

The Cartan form $\mathcal{L}^{(10|32)}$ splits under the decomposition of table 4 into vielbein and connection terms as

$$\begin{aligned}
\mathcal{L}^{(10|32)} = {} & \frac{1}{2}E^{\underline{\mu}}(P_{\underline{\mu}} + K_{\underline{\mu}}) + E^{\underline{Y}}D + E^{\underline{m}'}(\phi)\tilde{P}_{\underline{m}'} + \\
& + \bar{E}^Q_{\underline{a}} Q^{\underline{a}} + \bar{Q}_{\underline{a}} E^{\underline{a}}_Q + \bar{E}^S_{\underline{a}} S^{\underline{a}} + \bar{S}_{\underline{a}} E^{\underline{a}}_S + \frac{1}{2}\omega^{\underline{\mu}}(P_{\underline{\mu}} - K_{\underline{\mu}}) + \cdots \,,
\end{aligned} \tag{2.37}$$

where the omitted terms contain additional connection components. Using (2.36), we find, to lowest order in Y,

$$
\begin{aligned}
E^{\underline{\mu}} &= \frac{1}{Y}\left[dx^{\underline{\mu}} + \frac{1}{2}\left(d\bar\theta\gamma^{\underline{\mu}}\theta - \bar\theta\gamma^{\underline{\mu}}d\theta\right)\right] + O(Y) = e^{\underline{\mu}} + O(Y) \sim Y^{-1}\,,\\
E^{\underline{Y}} &= \frac{dY}{Y} + O(1) = e^{\underline{Y}} + O(1) \sim Y^{-1}\,,\\
E^{\underline{m}'} &= e^{\underline{m}'} + O(y) \sim O(1)\,,\\
E_Q^{\underline{a}} &= e^{\underline{a}} - Y^{1/2} e^{\underline{\mu}}\gamma_{\underline{\mu}}\vartheta^b u(\phi)_b{}^{\underline{a}} \sim Y^{-1/2}\,,\\
\bar E^Q_{\underline{a}} &= \bar e_{\underline{a}} + Y^{1/2} u^{-1}(\phi)_{\underline{a}}{}^b e^{\underline{\mu}}\bar\vartheta_b\gamma_{\underline{\mu}} \sim Y^{-1/2}\,.
\end{aligned}
\tag{2.38}
$$

Note that the bosonic vielbein, restricted to the boundary, is precisely that of $C_{(10|16)}$. The fermionic components receive corrections proportional to ϑ. Even so, e^Q is gauge-equivalent to E^Q via a coordinate-dependent $\mathbf{h}$-transformation of the form

$$
E^{\underline{M}} = e^{\underline{M}} + \left[\sqrt{Y}\left(\bar S^\alpha_a \vartheta^b_\alpha u_b{}^a + (u^{-1})_a{}^b \bar\vartheta^\alpha_b S^a_\alpha\right), e\,\right]^{\underline{M}}\,. \tag{2.39}
$$

2.4.2 From AdS superisometries to superconformal symmetries

With the tools and experience acquired in the previous subsections, we are now ready to prove the main result of this section: that the superisometries of the $AdS_5 \times S^5$ superspace

$$
C_{(10|32)} = \frac{\mathrm{SU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)} \tag{2.40}
$$

reduce near the boundary $Y = 0$ of AdS_5 to the superconformal transformations of the boundary conformal superspace

$$
C_{(10|16)} = \frac{\mathrm{SU}(2,2|4)}{\mathrm{Span}\,(\mathrm{iso}(1,3)_K \oplus \mathrm{so}(5) \oplus S)}\,. \tag{2.41}
$$

We choose the coset representatives (2.30) and (2.31), whose relation is given by (2.32) and (2.30).

Our proof proceeds in two steps. First, given a superisometry generator $\mathbf{g}$ in the Lie superalgebra of SU$(2,2|4)$, we find the compensating transformation $\mathbf{h}_{(10|32)}$ in the Lie algebra of the stability group SO$(1,4)\times$SO(5). In practice, we will not compute $\mathbf{h}_{(10|32)}$ in all its glory, but we will extract the properties that we will need — in particular, the Y-dependence. Given the requisite information about $\mathbf{h}_{(10|32)}$, our second step will be to show that $(h_{(10|16)}\mathbf{h}_{(10|32)}h^{-1}_{(10|16)})|_{C_{(10|16)}}$ approaches zero in the limit $Y \to 0$. This implies via (2.26) that the $C_{(10|32)}$ superisometries and the $C_{(10|16)}$ superconformal transformations agree on the boundary.

Let us recall from table 2 that the stability group $H_{(10|32)} \equiv \mathrm{SO}(1,4)\times\mathrm{SO}(5)$ is generated by $\frac{1}{2}(P_\mu - K_\mu)$, $M_{\mu\nu}$, and U_i^{Hj}, where U_i^{Hj} denotes the restriction of U_i^j to the generators of the stability group. The generators K_μ, $M_{\mu\nu}$ and U_i^{Hj} are shared by $H_{(10|16)} \equiv \mathrm{Span}(\mathrm{iso}(1,3|4)_K \oplus \mathrm{so}(5) \oplus S)$. Therefore, only the term in $\mathbf{h}_{(10|32)}$ proportional to the generator $\frac{1}{2}(P_\mu - K_\mu)$, and only its P_μ part, at that, can contribute to $(h_{(10|16)}\mathbf{h}_{(10|32)}h^{-1}_{(10|16)})|_{C_{(10|16)}}$: the other terms are projected out in restricting to the $C_{(10|16)}$ coset generators. We are thus relieved of the burden of calculating $\mathbf{h}_{(10|32)}$ in its entirety.

We would like to determine the Y-dependence of the P_μ part of $\mathbf{h}_{(10|32)}$, which, by the preceding remarks, is the same as the Y-dependence of its $\frac{1}{2}(P_\mu - K_\mu)$ piece. For this purpose, it is convenient to factor $G_{(10|32)}$ as the product of a term built from generators of weight > 0 and a term built from generators of weight ≤ 0,

$$c_{(10|32)} = g_+ g_- \,, \qquad g_+ = e^{x^\mu P_\mu} e^{\bar{Q}\theta + \bar{\theta}Q} \,, \qquad g_- = e^{\bar{S}\vartheta + \bar{\vartheta}S} u(\phi) Y^D \,. \tag{2.42}$$

The coset superisometry generated by $\mathbf{g}$ satisfies

$$\delta c_{(10|32)} = \mathbf{g} g_+ g_- - g_+ g_- \mathbf{h}_{(10|32)} \,. \tag{2.43}$$

Since g_+ is nothing but the coset representative of the space $C_{(4|16)}$, we may likewise write $\mathbf{g} g_+ = \delta_+ g_+ - g_+ \mathbf{h}_+$, where $\delta_+ g_+$ is the infinitesimal variation of g_+ generated by $\mathbf{g}$, and $\mathbf{h}_+$ is chosen to guarantee that $\delta_+ g_+$ lies in $C_{(4|16)}$. As we see from table 4, $\mathbf{h}_+$ is a linear combination of generators $\mathbf{g}_i$ of weights $w_i \leq 0$,

$$\mathbf{h}_+ = \sum_{i:w_i \leq 0} a_i \mathbf{g}_i \,. \tag{2.44}$$

Now, by the Baker-Campbell-Hausdorff formula,

$$g_+ \left(\sum_{i:w_i \leq 0} a_i \mathbf{g}_i \right) g_- = g_+ g_- \left(\sum_{i:w_i \leq 0} b_i Y^{-w_i} \mathbf{g}_i \right) , \tag{2.45}$$

for some new (Y-independent) coefficients b_i. Therefore,

$$\tilde{\delta} c_{(10|32)} \equiv \delta c_{(10|32)} - (\delta_+ g_+) g_- = g_+ g_- \left(- \sum_{i:w_i \leq 0} b_i Y^{-w_i} \mathbf{g}_i - \mathbf{h}_{(10|32)} \right) . \tag{2.46}$$

By construction, $\tilde{\delta} c_{(10|32)}$ is a variation such that $c_{(10|32)} + \tilde{\delta} c_{(10|32)} \in C_{(10|32)}$. We can see this because g_+ is only a function of x, θ, and $\bar{\theta}$ only; it does not depend on the other coordinates of $C_{(10|32)}$. The transformation $\mathbf{h}_{(10|32)}$ is chosen to compensate those terms in $g_+ g_- (\sum_{i:w_i \leq 0} b_i Y^{-w_i} \mathbf{g}_i)$ which pull $c_{(10|32)}$ away from the coset. Some of these terms may be proportional to K_μ. Since K_μ has weight -1, the coefficient of $\mathbf{h}_{(10|32)}|_{K_\mu} = \mathbf{h}_{(10|32)}|_{P_\mu - K_\mu} = \mathbf{h}_{(10|32)}|_{P_\mu}$ is proportional to Y.

Having established that $\mathbf{h}_{(10|32)}|_{P_\mu} \sim Y P_\mu$, it remains to calculate the behavior of $(h_{(10|16)}\mathbf{h}_{(10|32)}h^{-1}_{(10|16)})|_{C_{(10|16)}}$ near $Y = 0$. Straightforward manipulations give

$$\begin{aligned} h_{(10|16)}\, \mathbf{h}_{(10|32)}\big|_{P_\mu} h^{-1}_{(10|16)} &= Y^{-D} u^{-1}(\phi) e^{\bar{S}\vartheta + \bar{\vartheta}S} u(\phi) Y^D (Y P_\mu) \times \\ &\quad \times Y^{-D} u^{-1}(\phi) e^{-(\bar{S}\vartheta + \bar{\vartheta}S)} u(\phi) Y^D \\ &= Y^{-D} \left(u^{-1}(\phi) e^{\bar{S}\vartheta + \bar{\vartheta}S} u(\phi) (Y^2 P_\mu) u^{-1}(\phi) e^{-(\bar{S}\vartheta + \bar{\vartheta}S)} u(\phi) \right) Y^D \\ &= Y^{-D} (Y^2 \mathcal{O}) Y^D \,, \end{aligned} \tag{2.47}$$

where the operator $\mathcal{O}$ is a sum of (Y-independent) operators of weight 1 or lower. Therefore, $Y^{-D}(Y^2\mathcal{O})Y^D$ contains terms of the form $Y^{2-w_i}\mathcal{O}_i$, with $w_i \leq 1$. The resulting expression carries a positive power of Y, and therefore vanishes at the boundary $Y = 0$. By (2.26), this establishes the equality of the $C_{(10|32)}$ superisometries and the $C_{(10|16)}$ superconformal transformations at the boundary.

3. Kappa symmetry of the supersymmetric Wilson loop

In the last section, we studied the relation between the $AdS_5 \times S^5$ superspace $C_{(10|32)}$ and the conformal superspace $C_{(10|16)}$. We discovered that, just as in the bosonic case, the boundary coordinates of $C_{(10|16)}$ can be identified with a subset of the bulk coordinates of $C_{(10|32)}$. In the next two sections, we explore implications of this geometric fact for Wilson loops in the super-Yang-Mills theory on the boundary. In particular, we use the identification of coordinates between the two superspaces to show that the κ-symmetry of string worldsheets in type-IIB string theory on an $AdS_5 \times S^5$ background coincides with the κ-symmetry of the Wilson loops on which the string worldsheets terminate.

The Green-Schwarz string in the bulk naturally propagates in the superspace $C_{(10|32)}$ [6]. On the other hand, the role of $C_{(10|16)}$ is evident if we view $\mathcal{N} = 4$ super-Yang-Mills theory as being obtained by dimensional reduction from $\mathcal{N} = 1$ super-Yang-Mills theory in ten dimensions. The Wilson loop operator of this theory is defined as

$$\begin{aligned} W &= \frac{1}{N} \operatorname{tr} P \exp\left(\oint d\tau \, \dot{z}^M e_M^{\underline{M}} A_{\underline{M}} \right) \\ &= \frac{1}{N} \operatorname{tr} P \exp\left(\oint d\tau \left(A_{\underline{\hat{\mu}}}(z(\tau)) p^{\underline{\hat{\mu}}} + A_{\underline{\hat{\alpha}}}(z(\tau)) \dot{\lambda}^{\underline{\hat{\alpha}}}(\tau) \right) \right), \end{aligned} \tag{3.1}$$

where the trace is taken in the fundamental of SU(N). In this expression, $A_{\underline{M}} = (A_{\underline{\hat{\mu}}}, A_{\underline{\hat{\alpha}}})$ is the gauge superfield, written in the local Lorentz frame, and

$$p^{\underline{\hat{\mu}}} = \dot{x}^{\underline{\hat{\mu}}} + \frac{1}{2}\dot{\lambda}\Gamma^{\underline{\hat{\mu}}}\lambda - \frac{1}{2}\lambda\Gamma^{\underline{\hat{\mu}}}\dot{\lambda}\,, \tag{3.2}$$

where $\dot{} = d/d\tau$. By taking the connection $A_{\underline{M}}$ to be independent of the six extra coordinates y^m, we obtain a Wilson loop in the four-dimensional $\mathcal{N} = 4$ theory. For the remainder of this section, we will not underline tangent space indices (i.e., $\underline{\mu} \to \mu$), because we will be working entirely in tangent space.

The gauge superfield $(A_{\hat{\mu}}, A_{\hat{\alpha}})$ defines the covariant derivatives

$$\mathcal{D}_{\hat{\mu}} = \partial_{\hat{\mu}} + A_{\hat{\mu}}\,, \qquad \mathcal{D}_{\hat{\alpha}} = D_{\hat{\alpha}} + A_{\hat{\alpha}} \tag{3.3}$$

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on the superspace. The field strengths are defined by

$$
\begin{aligned}
F_{\hat{\alpha}\hat{\beta}} &= \{\mathcal{D}_{\hat{\alpha}}, \mathcal{D}_{\hat{\beta}}\} - 2\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}}\mathcal{D}_{\hat{\mu}}\,, \\
F_{\hat{\mu}\hat{\alpha}} &= [\mathcal{D}_{\hat{\mu}}, \mathcal{D}_{\hat{\alpha}}]\,, \\
F_{\hat{\mu}\hat{\nu}} &= [\mathcal{D}_{\hat{\mu}}, \mathcal{D}_{\hat{\nu}}]\,.
\end{aligned} \tag{3.4}
$$

We also use the spinor superfield defined by

$$
\Psi^{\hat{\alpha}} = \frac{1}{10}\Gamma^{\hat{\alpha}}_{\hat{\mu}\hat{\beta}}F^{\hat{\mu}\hat{\beta}}\,. \tag{3.5}
$$

In an expansion in fermionic coordinates, the leading terms of the superfields are given by the physical gauge field $a_{\hat{\mu}}(x)$ and the gaugino $\psi^{\hat{\alpha}}(x)$,

$$
A_{\hat{\mu}}(z) = a_{\hat{\mu}}(x) + O(\lambda)\,, \qquad \Psi^{\hat{\alpha}}(z) = \psi^{\hat{\alpha}} + O(\lambda)\,, \tag{3.6}
$$

and the subleading terms in the expansion contain auxiliary fields. We will discuss shortly how to remove these auxiliary fields. After dimensional reduction to four dimensions, the ten-dimensional gauge field $a_{\hat{\mu}}$ and the gaugino $\psi^{\hat{\alpha}}$ decompose as $a_{\hat{\mu}} = (a_\mu, \varphi_m)$ and $\psi^{\hat{\alpha}} = \psi^{\alpha a}$.

Strictly speaking, the ten-dimensional $\mathcal{N} = 1$ superspace is not identical to the conformal superspace $C_{(10|16)}$. The two spaces differ in the action of global supersymmetry on the coordinates y^m. If ϵ is the infinitesimal supersymmetry parameter, then

$$
\delta_\epsilon y^m = 0 \tag{3.7}
$$

in $C_{(10|16)}$ [2], whereas

$$
\delta_\epsilon y^m = \epsilon\,\Gamma^m\lambda \tag{3.8}
$$

in the $\mathcal{N} = 1$ superspace. For our purposes, this difference is not significant. After dimensional reduction to four dimensions, the Wilson loop (3.1) depends on $y^m(\tau)$ only through $\dot{y}^m$ in p^m. Therefore, if we redefine $p^{\hat{\mu}}$ as

$$
\begin{aligned}
p^\mu &= \dot{x}^\mu + \frac{1}{2}\dot{\lambda}\Gamma^\mu\lambda - \frac{1}{2}\lambda\Gamma^\mu\dot{\lambda}\,, \\
p^m &= \dot{y}^m\,,
\end{aligned} \tag{3.9}
$$

it is consistent to set $\delta_\epsilon y^m = 0$, as in $C_{(10|16)}$. In this section, when we study the properties of Wilson loops in super-Yang-Mills theory, we will work in the ten-dimensional $\mathcal{N} = 1$ superspace, since it simplifies our computation. In section 4, when we match the κ-symmetries of the Wilson loop and the string worldsheet, we will make the redefinition (3.9), and regard $z(\tau)$ as a loop in $C_{(10|16)}$.

The purpose of this section is to study the κ-invariance of the Wilson loop operator in the gauge theory. In section 3.1, we define κ-symmetry and prove that, if the loop is lightlike, W is classically κ-invariant. Then in section 3.2 we develop the technology needed for the quantum-mechanical version of the same result: that under the same hypotheses, the expectation value $\langle W\rangle$ is κ-invariant, to lowest order in a λ-expansion. The proof is provided in section 3.3, where we also comment on the case of loops with intersections.

3.1 Classical kappa invariance

We would like to find a κ-symmetry under which the Wilson loop (3.1) is invariant. A natural proposal for the κ-variations of the coordinates is

$$\begin{aligned} \delta_\kappa x^{\hat\mu} &= -\lambda\Gamma^{\hat\mu}\delta_\kappa\lambda\,, \\ \delta_\kappa\lambda &= \not{p}\kappa\,. \end{aligned} \tag{3.10}$$

These are the usual κ-invariances of the action of a superparticle. Acting with (3.10) on (3.1) yields

$$\delta_\kappa W = -\frac{1}{N}\,\mathrm{tr}\,P\oint d\tau\left(p^{\hat\mu}\delta_\kappa\lambda^{\hat\alpha}F_{\hat\mu\hat\alpha} + \delta_\kappa\lambda^{\hat\alpha}\dot\lambda^{\hat\beta}F_{\hat\alpha\hat\beta}\right)\exp\left(\oint d\tau' A_{\hat\mu}p^{\hat\mu} + A_{\hat\alpha}\dot\lambda^{\hat\alpha}\right). \tag{3.11}$$

The Wilson loop is κ-invariant, $\delta_\kappa W = 0$, if

$$F_{\hat\alpha\hat\beta} = 0\,, \tag{3.12}$$

and

$$p^2 = 0 \tag{3.13}$$

at every point on the loop. (Note that $p^{\hat\mu}(\tau)$ need not be constant in τ.) First of all, if $F_{\hat\alpha\hat\beta} = 0$, the term $\delta_\kappa\lambda^{\hat\alpha}\dot\lambda^{\hat\beta}F_{\hat\alpha\hat\beta}$ in (3.11) is manifestly zero. Moreover, if $p^2 = 0$, the term $p^{\hat\mu}\delta_\kappa\lambda^{\hat\alpha}F_{\hat\mu\hat\alpha}$ also vanishes. To see this, we start with the Jacobi identity

$$[\{\mathcal{D}_{\hat\alpha},\mathcal{D}_{\hat\beta}\},\mathcal{D}_{\hat\gamma}] + (\text{cyclic}) = 0\,. \tag{3.14}$$

Substituting in the definition of $F_{\hat\alpha\hat\beta}$ and employing the Dirac matrix identities listed in appendix A.3 gives

$$F_{\hat\mu\hat\alpha} = (10\Gamma_{\hat\mu}\Psi)_{\hat\alpha} - \frac{1}{40}\Gamma_{\hat\mu}^{\hat\beta\hat\gamma}\left(\mathcal{D}_{\hat\alpha}F_{\hat\beta\hat\gamma} + (\text{cyclic})\right), \tag{3.15}$$

where $\Psi^{\hat\alpha}$ is as in (3.5). The second term in the right-hand side of (3.15) is zero, as $F_{\hat\alpha\hat\beta} = 0$. The first term vanishes when multiplied by $p^{\hat\mu}\delta_\kappa\lambda^{\hat\alpha}$,

$$p^{\hat\mu}\delta_\kappa\lambda^{\hat\alpha}F_{\hat\mu\hat\alpha} = -10p^{\hat\mu}\delta_\kappa\lambda\Gamma^{\hat\mu}\Psi = -10\kappa\not{p}\not{p}\Psi = 0\,, \tag{3.16}$$

because $p^2 = 0$. Thus we have found that a Wilson loop W obeying $p^2 = 0$ everywhere is κ-invariant, provided the connection superfield obeys the condition $F_{\hat\alpha\hat\beta} = 0$.

What is the meaning of the condition $F_{\hat\alpha\hat\beta} = 0$? As is typical in superspace formulations of gauge theories, the gauge superfield A_M contains many auxiliary fields. To eliminate these fields in favor of the physical gauge field $a_{\hat{\mu}}(x)$ and gaugino $\psi^{\hat{\alpha}}(x)$, we must impose a constraint. The correct constraint is precisely [11]

$$F_{\hat{\alpha}\hat{\beta}} = 0\,. \tag{3.17}$$

Moreover, the $\mathcal{N} = 4$ theory in four dimensions has the special property that the constraint (3.17) not only eliminates the auxiliary fields, but also imposes the equations of motion

$$\begin{aligned} \nabla^{\hat{\mu}} f_{\hat{\mu}\hat{\nu}} + \frac{1}{2}\Gamma_{\hat{\nu}\hat{\alpha}\hat{\beta}}\{\psi^{\hat{\alpha}}, \psi^{\hat{\beta}}\} &= 0\,, \\ \nabla\!\!\!/\psi &= 0 \end{aligned} \tag{3.18}$$

of the gauge field and the gaugino [11, 12] ($\nabla_{\hat{\mu}} = D_{\hat{\mu}} + a_{\hat{\mu}}$; $f_{\hat{\mu}\hat{\nu}} = [\nabla_{\hat{\mu}}, \nabla_{\hat{\nu}}]$). The converse is also true: if the equations of motion are satisfied, then the constraint is automatically enforced. We will see this explicitly in the next subsection. This property is an inconvenience if one wants an off-shell supersymmetric formulation of the $\mathcal{N} = 4$ gauge theory, but for us, it turns out to be a blessing.

We may gain insight into the condition $p^2 = 0$ by recalling that our κ-symmetry variations are those of a massless superparticle. In [14], the action of a massless superparticle minimally coupled to a background U(1) gauge superfield was given. This action was shown to be κ-invariant if the gauge fields obey the equations of motion. If we use $p^2 = 0$, the kinetic term $\frac{1}{2}p^2$ vanishes, and what remains is just the exponent of the Wilson loop (3.1). It is therefore reasonable that W should be κ-invariant if $p^2 = 0$ and the gauge fields are on-shell.

Another interpretation of the condition $p^2 = 0$ was presented in [11, 12], where it was shown that the constraint $F_{\hat{\alpha}\hat{\beta}} = 0$ is equivalent to the condition that the gauge superfield be integrable on $(1|8)$-dimensional lightlike lines in superspace, with the 8 fermionic dimensions provided by the κ-transformation. Our application to Wilson loops relaxes the requirement that $p^{\hat{\mu}}(\tau)$ be constant.

3.2 Elimination of auxiliary fields

Equations of motion in a classical field theory lead to Schwinger-Dyson equations in its quantum counterpart. Since the super-Yang-Mills equations of motion imply $\delta_\kappa W = 0$ for a lightlike loop, we can use the associated Schwinger-Dyson equation to examine whether the vacuum expectation value $\langle W \rangle$ of the loop remains κ-invariant in the quantum theory. To compute $\delta_\kappa \langle W \rangle$ in practice, we must eliminate the auxiliary fields and express W in terms of the physical gauge field $a_{\hat{\mu}}$ and the gaugino ψ alone. A systematic procedure for doing this was introduced in [15].

As we have noted, the constraint $F_{\hat{\alpha}\hat{\beta}} = 0$ not only eliminates all auxiliary fields in favor of physical fields $a_{\hat{\mu}}$ and ψ, but also imposes the equations of motion, and only the equations of motion, on the physical fields. The procedure developed in [15] separates these two aspects of the constraint. Let us first discuss the elimination of the auxiliary fields. By combining the constraint (3.12) with the Bianchi identities of the gauge theory, it is possible to derive the relations [12]

$$\begin{aligned} F_{\hat\alpha\hat\beta} &= 0 \\ F_{\hat\mu\hat\alpha} - (\Gamma_{\hat\mu}\Psi)_{\hat\alpha} &= 0 \\ \mathcal{D}_{\hat\alpha}\Psi^{\hat\beta} - \frac{1}{2}\Gamma^{\hat\mu\hat\nu\hat\beta}_{\hat\alpha}F_{\hat\mu\hat\nu} &= 0 \\ \mathcal{D}_{\hat\alpha}F_{\hat\mu\hat\nu} + \mathcal{D}_{[\hat\mu,}(\Gamma_{\hat\nu]}\Psi)_{\hat\alpha} &= 0\,. \end{aligned} \tag{3.19}$$

Now we fix the fermionic gauge invariance. Following [15], we adopt the gauge-fixing condition

$$\lambda^{\hat\alpha}A_{\hat\alpha}(x,\lambda) = 0\,. \tag{3.20}$$

We define the operator

$$\mathbf{D} = \lambda^{\hat\alpha}\mathcal{D}_{\hat\alpha} = \lambda^{\hat\alpha}\partial_{\lambda^{\hat\alpha}}\,, \tag{3.21}$$

where the second equality is a consequence of the gauge-fixing condition (3.20). Multiplying the relations (3.19) by $\lambda^{\hat\alpha}$ and using the gauge-fixing condition leads to the **D**-*recursion relations*

$$\begin{aligned} (1+\mathbf{D})A_{\hat\alpha} &= 2(\Gamma^{\hat\mu}\lambda)_{\hat\alpha}A_{\hat\mu} \\ \mathbf{D}A_{\hat\mu} &= -\lambda\Gamma_{\hat\mu}\Psi \\ \mathbf{D}\Psi^{\hat\alpha} &= \frac{1}{2}(\lambda\Gamma^{\hat\mu\hat\nu})^{\hat\alpha}F_{\hat\mu\hat\nu} \\ \mathbf{D}F_{\hat\mu\hat\nu} &= \lambda\Gamma_{[\hat\mu,}\mathcal{D}_{\hat\nu]}\Psi\,. \end{aligned} \tag{3.22}$$

For example, the constraint $F_{\hat\alpha\hat\beta} = 0$ gives rise to the first equation in (3.22), since

$$\begin{aligned} 0 &= \lambda^{\hat\alpha}F_{\hat\alpha\hat\beta} \\ &= \lambda^{\hat\alpha}\{\mathcal{D}_{\hat\alpha},\mathcal{D}_{\hat\beta}\} - 2(\Gamma^{\hat\mu}\lambda)_{\hat\beta}\mathcal{D}_{\hat\mu} \\ &= (1+\lambda^{\hat\alpha}D_{\hat\alpha})A_{\hat\beta} - 2(\Gamma^{\hat\mu}\lambda)_{\hat\beta}A_{\hat\mu}\,. \end{aligned} \tag{3.23}$$

The operator **D** acts on a homogeneous polynomial in λ by multiplication by the degree of homogeneity; it does not change its degree. Therefore, the relations (3.22) are indeed recursive in powers of λ. They enable us to reconstruct the superfields $A_{\hat\mu}$, $A_{\hat\alpha}$, and $\Psi^{\hat\alpha}$ in their entirety from the lowest-order data

$$A_{\hat\mu} = a_{\hat\mu} + O(\lambda), \qquad A_{\hat\alpha} = O(\lambda)\,, \qquad \Psi^{\hat\alpha} = \psi^{\hat\alpha} + O(\lambda)\,. \tag{3.24}$$

The result is

$$\begin{aligned} A_{\hat\mu} &= a_{\hat\mu} - \lambda\Gamma_{\hat\mu}\psi - \frac{1}{4}(\lambda\Gamma_{\hat\mu}\Gamma^{\hat\nu\hat\rho}\lambda)\left(f_{\nu\rho} + \frac{2}{3}\Gamma_{[\hat\nu,}\mathcal{D}_{\hat\rho]}\psi\right) + \cdots \\ A_{\hat\alpha} &= (\Gamma^{\hat\mu}\lambda)_{\hat\alpha}\left[a_{\hat\mu} - \frac{2}{3}\lambda\Gamma_{\hat\mu}\psi - \frac{1}{8}(\lambda\Gamma_{\hat\mu}\Gamma^{\hat\nu\hat\rho}\lambda)f_{\hat\nu\hat\rho} + \cdots\right] \\ \Psi^{\hat\alpha} &= \psi^{\hat\alpha} + \frac{1}{2}(\Gamma^{\hat\mu\hat\nu}\lambda)^{\hat\alpha}(f_{\hat\mu\hat\nu} - \lambda\Gamma_{[\hat\mu,}\mathcal{D}_{\hat\nu]}\psi) + \cdots\,. \end{aligned} \tag{3.25}$$

The superfields in (3.25) are written exclusively in terms of the physical gauge field a_μ, its field strength $f_{\mu\nu}$, and the gaugino ψ; all auxiliary fields have been eliminated. Moreover, the gauge-fixing condition (3.20) is automatically satisfied. This is because the lowest-order data and the first equation in (3.22) imply that $A_{\hat\alpha} = (\lambda\Gamma_{\hat\mu})_{\hat\alpha} V^{\hat\mu}$, for some vector $V^{\hat\mu}$. The condition $\lambda^{\hat\alpha} A_{\hat\alpha} = 0$ then follows from the identity $\lambda\Gamma^{\hat\mu}\lambda = 0$, which is a consequence of the symmetry of the Dirac matrices, $\Gamma^{\hat\mu}_{\hat\alpha\hat\beta} = \Gamma^{\hat\mu}_{\hat\beta\hat\alpha}$.

We have not yet exhausted all the relations that follow from the constraint $F_{\hat\alpha\hat\beta} = 0$. Our next task is to show that, if we substitute the solution (3.25) of the **D**-recursion relations into the constraint, we obtain the equations of motion for $a_{\hat\mu}$ and ψ, and nothing else. It is possible to calculate $F_{\hat\alpha\hat\beta}$ from (3.25) by brute force. To the first few orders in λ, we find

$$F_{\hat\alpha\hat\beta} = (\Gamma_{\hat\mu}\lambda)_{\hat\alpha}(\Gamma_{\hat\nu}\lambda)_{\hat\beta}\left[-\frac{2}{15}\lambda\Gamma^{\hat\mu\hat\nu}\nabla\!\!\!/\psi - \frac{1}{18}(\lambda\Gamma^{\hat\mu\hat\nu}\Gamma^{\hat\rho}\lambda)\left(\nabla^{\hat\sigma} f_{\hat\sigma\hat\rho} + \frac{1}{2}\Gamma_{\hat\rho\hat\gamma\hat\delta}\{\psi^{\hat\gamma},\psi^{\hat\delta}\}\right) + \cdots\right]. \tag{3.26}$$

This expansion expresses $F_{\hat\alpha\hat\beta}$ in terms of quantities that are set to zero by the super-Yang-Mills equations of motion (3.18). Its explicit form will be used in the next subsection to evaluate the Schwinger-Dyson equation.

Calculating even the first terms of the $F_{\hat\alpha\hat\beta}$ expansion in this manner calls for algebraic heroics. A more workable method relies on a set of **D**-recursions for the relations (3.19). The Bianchi identities and the Dirac matrix properties listed in appendix A.3 imply

$$\begin{aligned}
(2+\mathbf{D})F_{\hat\alpha\hat\beta} &= 2(\Gamma^{\hat\mu}\lambda)_{\hat\alpha}\left(F_{\hat\mu\hat\beta} - (\Gamma_{\hat\mu}\Psi)_{\hat\beta}\right) + (\hat\alpha\leftrightarrow\hat\beta)\,,\\
(1+\mathbf{D})\left(F_{\hat\mu\hat\alpha} - (\Gamma_{\hat\mu}\Psi)_{\hat\alpha}\right) &= (\Gamma_{\hat\mu}\lambda)_{\hat\beta}\left(\mathcal{D}_{\hat\alpha}\Psi^{\hat\beta} - \frac{1}{2}\Gamma^{\hat\mu\hat\nu\hat\beta}_{\hat\alpha} F_{\hat\mu\hat\nu}\right),\\
(1+\mathbf{D})\left(\mathcal{D}_{\hat\alpha}\Psi^{\hat\beta} - \frac{1}{2}\Gamma^{\hat\mu\hat\nu\hat\beta}_{\hat\alpha}F_{\hat\mu\hat\nu}\right) &= (\Gamma^{\hat\mu\hat\nu}\lambda)^{\hat\beta}\mathcal{D}_{\hat\mu}\left(F_{\hat\nu\hat\alpha} - (\Gamma_{\hat\nu}\Psi)_{\hat\alpha}\right) + (\Gamma^{\hat\mu}\lambda)_{\hat\alpha}(\Gamma_{\hat\mu}\mathcal{D}\!\!\!\!/\,\Psi)^{\hat\beta} -\\
&\quad - \delta^{\hat\beta}_{\hat\alpha}(\lambda\mathcal{D}\!\!\!\!/\,\Psi) - \lambda^{\hat\beta}(\mathcal{D}\!\!\!\!/\,\Psi)_{\hat\alpha}\,.
\end{aligned} \tag{3.27}$$

This chain of equations relates $F_{\hat\alpha\hat\beta}$ and $F_{\hat\mu\hat\alpha}$ to the equation of motion for the superfield Ψ. To express $F_{\hat\alpha\hat\beta}$ and $F_{\hat\mu\hat\alpha}$ in terms of the component field equations of motion, we use the recursion relations

$$\begin{aligned}
\mathbf{D}(\mathcal{D}\!\!\!\!/\,\Psi)_{\hat\alpha} &= (\Gamma^{\hat\mu}\lambda)_{\hat\alpha}\left(\mathcal{D}^{\hat\nu}F_{\hat\nu\hat\mu} + \frac{1}{2}\Gamma_{\hat\mu\hat\beta\hat\gamma}\{\Psi^{\hat\beta},\Psi^{\hat\gamma}\}\right)\\
\mathbf{D}\left(\mathcal{D}^{\hat\nu}F_{\hat\nu\hat\mu} + \frac{1}{2}\Gamma_{\hat\mu\hat\alpha\hat\beta}\{\Psi^{\hat\alpha},\Psi^{\hat\beta}\}\right) &= \lambda\Gamma_{\hat\mu\hat\nu}\mathcal{D}^{\hat\nu}\mathcal{D}\!\!\!\!/\,\Psi\,,
\end{aligned} \tag{3.28}$$

which are also derived from the Bianchi identities and the Dirac algebra. Starting with the initial condition (3.24), we can solve (3.27) and (3.28) iteratively in powers of λ, to express $F_{\hat\alpha\hat\beta}$ and $F_{\hat\mu\hat\alpha}$ in terms of $a_{\hat\mu}$, ψ and their derivatives. This procedure generates the λ-expansion (3.26). Apart from being calculationally tractable, it provides a general proof that the constraint $F_{\hat{\alpha}\hat{\beta}} = 0$ implies the super-Yang-Mills equations of motion and nothing more.

3.3 The Schwinger-Dyson equation

Given our algorithm for expressing W in terms of the physical fields $a_{\hat{\mu}}$ and ψ, we can apply the Schwinger-Dyson equation to compute $\delta_\kappa \langle W \rangle$. We wish to evaluate

$$\delta_\kappa \langle W \rangle = -\,\mathrm{tr}\, P \oint d\tau \left\langle \left(p^{\hat{\mu}} \delta_\kappa \lambda^{\hat{\alpha}} F_{\hat{\mu}\hat{\alpha}} + \delta_\kappa \lambda^{\hat{\alpha}} \dot{\lambda}^{\hat{\beta}} F_{\hat{\alpha}\hat{\beta}} \right) W_0 \right\rangle , \tag{3.29}$$

where $W_0 = \frac{1}{N} \exp \left(\oint d\tau \left(A_\mu p^\mu + A_\alpha \dot{\lambda}^\alpha \right) \right)$. The recursion relations (3.27) and (3.28) give $F_{\hat{\mu}\hat{\alpha}}$ and $F_{\hat{\alpha}\hat{\beta}}$ in terms of

$$\not{\nabla} \psi = -g^2 \frac{\delta S_{SYM}}{\delta \psi} \tag{3.30}$$

and

$$\nabla^{\hat{\sigma}} f_{\hat{\sigma}\hat{\rho}} + \frac{1}{2} \Gamma_{\hat{\rho}\hat{\gamma}\hat{\delta}} \{ \psi^{\hat{\gamma}}, \psi^{\hat{\delta}} \} = g^2 \frac{\delta S_{SYM}}{\delta a_{\hat{\rho}}} , \tag{3.31}$$

where S_{SYM} is the action for the $\mathcal{N} = 4$ super-Yang-Mills theory, and g is the Yang-Mills coupling constant. An integration by parts in the functional integral transfers the functional derivatives $\delta/\delta\psi$ and $\delta/\delta a_{\hat{\mu}}$ onto W_0. Substituting the expansions (3.25) of the gauge superfields into the definition of W_0 then enables us to write the functional derivatives of W_0 in terms of the physical gauge field and gaugino. The entire procedure may be carried out to any desired order in λ.

Let us fill in some of the details of the calculation at lowest nontrivial order in λ. At this order, the $F_{\hat{\alpha}\hat{\beta}}$ term in (3.29) can be neglected, and the expansion of $F_{\hat{\mu}\hat{\alpha}}$ can be truncated at quadratic order in λ. From (3.15) and (3.26) we calculate

$$\begin{aligned} F_{\hat{\mu}\hat{\alpha}} = \frac{1}{300} \Big(& (\lambda \Gamma_{\hat{\lambda}} \Gamma_{\hat{\mu}} \Gamma_{\hat{\rho}} \lambda)(\Gamma^{\hat{\lambda}\hat{\rho}} \not{\nabla}\psi)_{\hat{\alpha}} - 4 (\lambda \Gamma_{\hat{\lambda}} \Gamma_{\hat{\mu}} \Gamma_{\hat{\rho}})_{\hat{\alpha}} (\lambda \Gamma^{\hat{\lambda}\hat{\rho}} \not{\nabla}\psi) + \\ & + 2\, \mathrm{tr}(\Gamma_{\hat{\mu}} \Gamma_{\hat{\lambda}}) (\lambda \Gamma_{\hat{\rho}})_{\hat{\alpha}} (\lambda \Gamma^{\hat{\lambda}\hat{\rho}} \not{\nabla}\psi) - 2 (\lambda \Gamma_{\hat{\rho}})_{\hat{\alpha}} (\lambda \Gamma_{\hat{\lambda}} \Gamma_{\hat{\mu}} \Gamma^{\hat{\lambda}\hat{\rho}} \not{\nabla}\psi) \Big) . \end{aligned} \tag{3.32}$$

Substituting this expression into (3.29) and applying Dirac matrix identities, we find

$$\begin{aligned} \delta_\kappa \langle W \rangle = -\frac{1}{300} \oint d\tau\ \Big\langle \Big(& (\lambda \Gamma_{\hat{\mu}} \not{p} \Gamma_{\hat{\nu}} \lambda)(\kappa \not{p} \Gamma^{\hat{\mu}\hat{\nu}} \not{\nabla}\, \psi) - 4 (\kappa \not{p} \Gamma_{\hat{\mu}} \not{p} \Gamma_{\hat{\nu}} \lambda)(\lambda \Gamma^{\hat{\mu}\hat{\nu}} \not{\nabla}\, \psi) + \\ & + 64 (\kappa \Gamma_{\hat{\mu}} \lambda)(\lambda [\not{p}, \Gamma^{\hat{\nu}}] \not{\nabla}\psi) - 2 (\kappa \not{p} \Gamma_{\hat{\mu}} \lambda)(\lambda \Gamma_{\hat{\nu}} \not{p} \Gamma^{\hat{\nu}\hat{\mu}} \not{\nabla}\psi) \Big) W_0 \Big\rangle . \end{aligned} \tag{3.33}$$

Each $\not{\nabla}\psi(x(\tau))$ is then replaced by the functional variation

$$\langle \not{\nabla}\psi(x(\tau)) W_0 \rangle = g^2 \frac{\delta \langle W_0 \rangle}{\delta \psi^{\hat{\alpha}}(x(\tau))} = g^2 \oint d\tau'\, \delta(x(\tau) - x(\tau')) (\not{p} \lambda)^{\hat{\alpha}} (\tau') \langle W_0 \rangle . \tag{3.34}$$

Each of the resulting terms on the right-hand side of (3.33) is zero. For example,

$$\begin{aligned}(\kappa\not{p}\Gamma_{\hat{\mu}}\lambda)(\lambda\Gamma_{\hat{\nu}}\not{p}\Gamma^{\hat{\nu}\hat{\mu}}\not{p}\lambda) &= (\kappa\not{p}\Gamma_{\hat{\mu}}\lambda)(\lambda\Gamma_{\hat{\nu}}\not{p}\Gamma^{\hat{\nu}}\Gamma^{\hat{\mu}}\not{p}\lambda) - (\kappa\not{p}\Gamma_{\hat{\mu}}\lambda)(\lambda\Gamma^{\hat{\mu}}\not{p}\not{p}\lambda) \\ &= -8(\kappa\not{p}\Gamma_{\hat{\mu}}\lambda)(\lambda\not{p}\Gamma^{\hat{\mu}}\not{p}\lambda) = 0\,, \end{aligned} \tag{3.35}$$

by $p^2 = 0$ and $\lambda\Gamma^{\hat{\mu}}\lambda = 0$. We conclude that, if the loop $x(\tau)$ is smooth and has no nontrivial self-intersections (i.e., if $x(\tau) = x(\tau')$ implies $\tau = \tau'$), then the vacuum expectation value of the Wilson loop is κ-invariant,

$$\delta_\kappa\langle W\rangle = 0\,. \tag{3.36}$$

The situation is different when the loop has a self-intersection point. In this case, (3.33) yields

$$\delta_\kappa\langle W\rangle = \frac{g^2N}{6}\oint d\tau_1\oint d\tau_2\delta(x(\tau_1)-x(\tau_2))p^{\hat{\mu}}(\tau_1)p^{\hat{\nu}}(\tau_2)(\lambda\Gamma_{\hat{\mu}\hat{\nu}\hat{\rho}}\lambda)(\kappa\not{p}(\tau_1)\Gamma^{\hat{\rho}}\lambda)\langle W_1W_2\rangle\,, \tag{3.37}$$

at leading nontrivial order in the λ-expansion, where W_1 and W_2 are operators associated to the loops obtained by recombining the original loop at the intersection.

The integrand is nonzero unless $p(\tau_1)$ is parallel to $p(\tau_2)$. Thus the κ-invariance of the Wilson loop is broken at self-intersection points.

4. Matching the Wilson loop to the string worldsheet

The purpose of this section is to show how the κ-invariance of Wilson loops can be understood, via the AdS/CFT correspondence, as following from κ-invariance of string theory in $AdS_5\times S^5$ superspace [6].

4.1 The Wilson loop as a boundary condition

According to the AdS/CFT correspondence, the expectation value of a Wilson loop operator in strongly coupled four-dimensional $\mathcal{N}=4$ super-Yang-Mills theory can be calculated from the worldsheet of type-IIB string theory on $AdS_5\times S^5$ [3, 4]. The super-Yang-Mills theory is thought of as living on the boundary of AdS_5, and the worldsheet is characterized by the requirement that it end on the Wilson loop. To make sense of this conjecture even for bosonic Wilson loops, it is necessary to think carefully about the boundary conditions the Wilson loop imposes on the string worldsheet ending on it. This subsection is devoted to reviewing these considerations and extending them to the supersymmetric case.

The supersymmetric Wilson loop is defined along a contour $z^M(\tau)$ in the conformal superspace $C_{(10|16)}$ with worldline parameter τ; that is,

$$z^M(\tau) = (x^\mu(\tau), y^m(\tau), \lambda^a_\alpha(\tau))\,. \tag{4.1}$$

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According to the conjecture, the loop is the boundary $\sigma = 0$ of a string worldsheet embedded in the $AdS_5 \times S^5$ superspace $C_{(10|32)}$, with worldsheet parameters (τ, σ) and embedding coordinates

$$Z^{\mathbf{M}}(\tau,\sigma) = (X^{\mu}(\tau,\sigma), Y^{m}(\tau,\sigma), \theta^{a}_{\alpha}(\tau,\sigma), \vartheta^{a}_{\alpha}(\tau,\sigma))\,. \tag{4.2}$$

The boundary conditions imposed by the Wilson loop on the string worldsheet must therefore translate into relations among the $z^M(\tau)$ and $Z^{\mathbf{M}}(\tau,\sigma)$ in the limit $\sigma \to 0$.

For the bosonic variables, the analysis proceeds exactly as in [10], where purely bosonic Wilson loops were considered. It is natural to impose Dirichlet conditions on X^μ,

$$X^{\mu}(\tau, \sigma = 0) = x^{\mu}(\tau)\,, \tag{4.3}$$

identifying the X^μ at the boundary of the worldsheet with the coordinates in the $\mathcal{N} = 4$ gauge theory. The relation between the y^m and Y^m coordinates is more subtle. It was argued in [10] that the appropriate boundary conditions on the Y^m are Neumann,

$$P^{m}_{\tau}(\tau, \sigma = 0) = \dot{y}^{m}(\tau)\,, \tag{4.4}$$

where we have introduced the conjugate momentum

$$P^{i}_{\hat{\mu}} = \frac{\delta \mathcal{L}}{\delta(\partial_i X^{\hat{\mu}})} \qquad (i = \tau, \sigma)\,, \tag{4.5}$$

and $\mathcal{L}$ is the string lagrangian. The momentum satisfies

$$P^{\hat{\mu}}_{i} = J^{j}_{i} E^{\hat{\mu}}_{j}\,, \tag{4.6}$$

where

$$J^{j}_{i} = \frac{g_{ik}\epsilon^{kj}}{\sqrt{g}} \tag{4.7}$$

is the worldsheet complex structure, written in terms of the worldsheet metric g and the antisymmetric tensor density ϵ^{ij} ($\epsilon^{\tau\sigma} = +1$).

Now to the fermionic variables. Our proposed boundary conditions for the fermions stem from the result, proven in section 2, that at the boundary $Y = 0$ of $AdS_5 \times S^5$ superspace, the ϑ coordinates decouple from the superisometry variations of the remaining coordinates. Moreover, the variations of the θ's reduce at the boundary to the variations under superconformal transformations of the λ's of the $(10|16)$-dimensional conformal superspace. This strongly suggests the boundary condition[3]

$$\theta^{a}_{\alpha}(\tau, \sigma = 0) = \lambda^{a}_{\alpha}(\tau)\,. \tag{4.8}$$

[3]This equation is not precise: as discussed in appendix A.2, the spinor θ_α is chiral under the Lorentz group $\mathrm{SO}(1,3)$ of the boundary, whereas λ_α, as defined in section 3, is Majorana. This distinction will not matter for us until we compare κ-symmetries in section 4.2.

What conditions must we impose at $Y = 0$ on the ϑ's? We claim that no further conditions are necessary. This may be understood from general features of the equations of motion for the string worldsheet in $AdS_5 \times S^5$ superspace [6]. The equations for the bosonic coordinates are second-order; those for the fermions are first-order; and the entire system is of elliptic type. A second-order elliptic system in a given region is completely specified by giving one piece of boundary data (i.e., a Dirichlet or Neumann condition) per variable, and indeed we found that the six Neumann conditions (4.4) and four Dirichlet conditions (4.3) are just sufficient to determine the ten bosonic worldsheet coordinates $X^\mu(\tau, \sigma)$ and $Y^m(\tau, \sigma)$. On the other hand, since the 32 fermionic equations are first-order, and two first-order equations are generically equivalent to one second-order equation, we expect that only 16 boundary conditions are required to determine all 32 fermionic coordinates θ and ϑ. Having already supplied 16 boundary conditions in (4.8), we need provide no further boundary data: $\theta(\tau, \sigma)$ and $\vartheta(\tau, \sigma)$ are uniquely fixed by the string equations of motion and the boundary values of $\theta(\tau, \sigma = 0)$. This point of view is consistent with the boundary decoupling of ϑ discovered in section 2.

Of course, this counting cannot be the whole story, since we have ignored κ-symmetry. Though the string worldsheet propagates in a superspace of 32 fermionic dimensions, 16 of these degrees of freedom are gauge artifacts, removable by fixing κ-symmetry. As we shall show in the next subsection, the 16-dimensional κ-symmetry decomposes in such a way that eight independent symmetries act on the θ coordinates and eight on the ϑ's. Thus, if we fix κ-symmetry entirely, we are left with 16 independent fermionic degrees of freedom: eight θ's and eight ϑ's. By the argument of the last paragraph, these are completely determined everywhere on the worldsheet by the values at the boundary $\sigma = 0$ of the eight "unfixed" θ's. This is consistent with our expectations from super-Yang-Mills theory. The worldline of a massless particle in ten dimensions coupled to a background gauge field is apparently $(1|16)$-dimensional, but this system, too, admits a κ-symmetry [14], which cuts the number of effective fermionic degrees of freedom to eight. At the boundary of the string worldsheet, the eight "unfixed" θ's may be identified with these eight λ's unfixed by κ-symmetry of the Wilson loop worldline.

The arguments we have just given, together with arguments made in [10] for the bosonic variables, demonstrate that the boundary conditions (4.3), (4.4), and (4.8) suffice to determine a minimal supersurface in $AdS_5 \times S^5$ superspace. However, the prescription of [3, 4, 10] demands in addition that the boundary $\sigma = 0$ of the worldsheet end on the boundary $Y^m = 0$ of AdS_5. This is not guaranteed from the boundary conditions alone. We now show that a necessary condition for the worldsheet to terminate on the boundary of AdS_5 is the condition $p^2 = 0$ on the loop variable. More specifically, we show that the Virasoro constraint of the string, evaluated at the boundary of the worldsheet, is equivalent to the condition $p^2 = 0$, up to terms that vanish at $Y = 0$.

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It is convenient to write the Virasoro constraints as

$$
\begin{aligned}
0 &= 2E_i^{\hat{\underline{\mu}}} E_{j\hat{\underline{\mu}}} - g_{ij} g^{kl} E_k^{\hat{\underline{\mu}}} E_{l\hat{\underline{\mu}}} \\
&= P_i \cdot P_j + E_i \cdot E_j \qquad (i, j = \tau, \sigma)\,,
\end{aligned} \tag{4.9}
$$

where we have used the identity

$$
\frac{1}{g} \epsilon^{ij} \epsilon^{kl} = g^{ik} g^{jl} - g^{il} g^{jk}. \tag{4.10}
$$

We will show in the following paragraphs that

$$
\frac{P_\tau^{\underline{\mu}} P_{\tau\underline{\mu}}}{E_\tau^{\underline{\mu}} E_{\tau\underline{\mu}}} = 0 \qquad \text{and} \qquad \frac{E_\tau^{\underline{m}} E_{\tau\underline{m}}}{P_\tau^{\underline{m}} P_{\tau\underline{m}}} = 0\,, \tag{4.11}
$$

in the limit $Y \to 0$. Granting this, the Virasoro constraints reduce to the single equation

$$
P_\tau^{\underline{m}} P_{\tau\underline{m}} + E_\tau^{\underline{\mu}} E_{\tau\underline{\mu}} = 0 \tag{4.12}
$$

at the boundary. This equation is equivalent to the condition $p^2 = 0$ on the Wilson loop. To see this, we use the form of the vielbeins given in section 2.4 and the boundary conditions on the string worldsheet to rewrite (4.12) as

$$
P_\tau^{\underline{m}} P_{\tau\underline{m}} + E_\tau^{\underline{\mu}} E_{\tau\underline{\mu}} = Y^{-2} \left[\left(\dot{x}^{\underline{\mu}} + \frac{1}{2}(\dot{\bar{\lambda}} \gamma^{\underline{\mu}} \lambda) - \frac{1}{2}(\bar{\lambda} \gamma^{\underline{\mu}} \dot{\lambda}) \right)^2 + \dot{y}^2 \right] + O(1) = 0\,. \tag{4.13}
$$

We then compare this with

$$
\begin{aligned}
p^{\underline{\mu}} &= \dot{x}^{\underline{\mu}} + \frac{1}{2} \dot{\bar{\lambda}} \Gamma^{\underline{\mu}} \lambda - \frac{1}{2} \bar{\lambda} \Gamma^{\underline{\mu}} \dot{\lambda}, \\
p^{\underline{m}} &= \dot{y}^{\underline{m}},
\end{aligned} \tag{4.14}
$$

where we have taken into account the redefinition of $\dot{y}$ discussed at the beginning of section 3. Working in the "5+5" basis for the Γ-matrices described in appendix A.1, we find

$$
p^2 = \left(\dot{x}^{\underline{\mu}} + \frac{1}{2} \dot{\bar{\lambda}} \gamma^{\underline{\mu}} \lambda - \frac{1}{2} \bar{\lambda} \gamma^{\underline{\mu}} \dot{\lambda} \right)^2 + \dot{y}^2\,. \tag{4.15}
$$

Upon multiplication by $1/Y^2$, this is identical to the leading term in (4.13). Thus the Virasoro condition of the string, restricted to the boundary, is equivalent to the condition $p^2 = 0$.

We now prove (4.11), assuming smooth worldsheet boundary conditions. Let us first describe the idea of the proof. We are interested in understanding how the worldsheet behaves as it approaches the boundary at $Y^m = 0$. If the worldsheet boundary is constrained to be bosonic and straight,

$$
\begin{aligned}
\lambda(\tau, \sigma = 0) &= 0 \\
x^{\hat{\mu}}(\tau, \sigma = 0) &= v^{\hat{\mu}} \tau\,,
\end{aligned} \tag{4.16}
$$

then the worldsheet itself must be approximately flat near the boundary and perpendicular to it: fluctuations in the worldsheet geometry are energetically costly, on account of the factor Y^{-2} in the AdS_5 metric. We claim that, no matter what the worldsheet boundary condition actually is, the worldsheet near any given point $\tau = \tau_0$ on the boundary can be locally approximated by a worldsheet obeying (4.16).

Let us now make this precise. Suppose (X^μ, Y^m, θ) is a solution of the string equations of motion with general boundary conditions

$$\begin{aligned} X^\mu(\tau, \sigma = 0) &= v^\mu \tau + \frac{1}{2} a^\mu \tau^2 + \cdots \\ P^m_\tau(\tau, \sigma = 0) &= v^m + a^m \tau + \cdots \\ \theta(\tau, \sigma = 0) &= \lambda_1 \tau + \lambda_2 \tau^2 + \cdots \\ Y^m(\tau, \sigma = 0) &= 0. \end{aligned} \tag{4.17}$$

We have set $\tau_0 = 0$ for simplicity. In addition, by suitably positioning the origin of the boundary coordinates, we have assumed $x^\mu = \lambda = 0$ at $\tau = 0$. Since the string worldsheet action in AdS_5 is scale-invariant, $(\widetilde{X}^\mu, \widetilde{Y}, \widetilde{\theta}) = (\Omega X^\mu, \Omega Y, \Omega^{1/2}\theta)$ is also a solution, with the boundary conditions

$$\begin{aligned} \widetilde{X}^\mu(\tau, \sigma = 0) &= \Omega v^\mu \tau + \frac{1}{2} \Omega a^\mu \tau^2 + \cdots \\ \widetilde{P}^m_\tau(\tau, \sigma = 0) &= \Omega v^m + \Omega a^m \tau + \cdots \\ \widetilde{\theta}(\tau, \sigma = 0) &= \Omega^{1/2} \lambda_1 \tau + \Omega^{1/2} \lambda_2 \tau^2 + \cdots \\ \widetilde{Y}^m(\tau, \sigma = 0) &= 0\,. \end{aligned} \tag{4.18}$$

The action moreover is invariant under worldsheet reparametrizations. The reparametrization $\tau \to \hat{\tau} = \Omega \tau$ transforms the boundary conditions to

$$\begin{aligned} \widetilde{X}^\mu(\hat{\tau}, \sigma = 0) &= v^\mu \hat{\tau} + \frac{1}{2} a^\mu \frac{\hat{\tau}^2}{\Omega} + \cdots \\ \widetilde{P}^m_{\hat{\tau}}(\hat{\tau}, \sigma = 0) &= v^m + a^m \frac{\hat{\tau}}{\Omega} + \cdots \\ \widetilde{\theta}(\hat{\tau}, \sigma = 0) &= \Omega^{-1/2} \lambda_1 \hat{\tau} + \Omega^{-3/2} \lambda_2 \hat{\tau}^2 + \cdots \\ \widetilde{Y}^m(\tau, \sigma = 0) &= 0\,. \end{aligned} \tag{4.19}$$

The expressions in (4.11) are invariant under both the rescaling of (X^μ, Y^m, θ) and the reparametrization of τ, for any Ω. In the limit $\Omega \to \infty$, the boundary conditions (4.19) become those of a straight and bosonic worldsheet,

$$\begin{aligned} \hat{X}^\mu(\hat{\tau}, \sigma = 0) &= v^\mu \hat{\tau} \\ \hat{P}^m_\tau(\hat{\tau}, \sigma = 0) &= v^m \\ \hat{\theta}(\hat{\tau}, \sigma = 0) &= 0 \\ \hat{Y}^m(\hat{\tau}, \sigma = 0) &= 0\,. \end{aligned} \tag{4.20}$$

It is therefore sufficient to establish (4.11) for worldsheets obeying the simple boundary conditions (4.20). First of all, since $Y^m = 0$ at the boundary, clearly $E_\tau^{\underline{m}} \sim \partial_\tau Y^{\underline{m}} = 0$. Furthermore, it was shown in [3] that worldsheets obeying (4.20) satisfy

$$X \sim Y^3 \to 0 \tag{4.21}$$

near the boundary. Computing J_i^j and $E_i^{\underline{\mu}}$ is then straightforward, and shows that $P_\tau^{\underline{\mu}} = 0$ at the boundary.

We have shown that, for a smooth Wilson loop, the condition $p^2 = 0$ is necessary in order for the string worldsheet to end on the boundary $Y = 0$. It is interesting to note that, for the same reason as found in [10], the agreement between the $p^2 = 0$ condition of the loop and the Virasoro constraints of the string worldsheet fails when the loop has intersections. The disagreement between the $p^2 = 0$ condition and the Virasoro constraint at these points may be a cause of the breakdown (3.37) of κ-invariance.

4.2 Matching the kappa symmetries

The prescription of the previous subsection implies that the Wilson loop expectation value $\langle W \rangle$ is obtained as a functional integral over string worldsheets obeying the boundary conditions (4.3), (4.4) and (4.8). We may thus view $\langle W \rangle$ as a wave function of Hartle-Hawking type, with no boundary other than the Wilson loop itself. Since the action of [6] is invariant under κ-symmetry, this wave function must obey a set of constraints, corresponding to the vanishing of the momenta conjugate to the directions of the κ-symmetry. This is a standard statement in any gauge theory. In Maxwell theory, for example, the momentum conjugate to the timelike component A_0 of the gauge field vanishes, and the wave function Ψ of the theory obeys the constraint $\delta\Psi/\delta A_0 = 0$. We claim that the constraint due to the κ-symmetry of the worldsheet is nothing but the equation

$$\delta_\kappa \langle W \rangle = 0 \tag{4.22}$$

obtained in section 3 within gauge theory.

To show that the constraint from the worldsheet κ-symmetry is the same as (4.22), it is sufficient to check that the κ-symmetries of the string worldsheet reduce at the boundary to the worldline κ-symmetries of the Wilson loop. The κ-transformations of the string propagating in the $AdS_5 \times S^5$ superspace given in [6] read

$$\begin{aligned} \delta_\kappa Z^{\mathbf{M}} E_{\mathbf{M}}^I &= 2 E^{\underline{\hat\mu}}{}_i \hat\Gamma_{\underline{\hat\mu}} \kappa^{Ii}\,, \\ \delta_\kappa Z^{\mathbf{M}} E_{\mathbf{M}}^{\underline{\hat\mu}} &= 0\,. \end{aligned} \tag{4.23}$$

Here $I = 1, 2$ labels the two Majorana-Weyl fermionic generators of type-IIB supergravity on $AdS_5 \times S^5$. The corresponding fermionic vielbeins $E^{1,2}$ are related to the fermionic vielbeins defined in section 2 by

$$E_{Q/S} = \frac{1}{2\sqrt{2}}\left(1 \mp \gamma^5\right)\left(E^1 + iE^2\right). \tag{4.24}$$

The $\hat{\Gamma}$-matrices are the ones defined in appendix A.1. The κ-symmetry parameters are packaged in two Majorana-Weyl quantities κ^{1i} and κ^{2i}, each of which carries a (hidden) spinor index as well as a (visible) worldsheet vector index $i = (\tau, \sigma)$. The κ^{Ii} obey the worldsheet self-duality relations

$$\begin{aligned} \kappa^1 &\equiv \kappa^{1\tau} = J_i^\tau \kappa^{1i} \\ \kappa^2 &\equiv \kappa^{2\tau} = -J_i^\tau \kappa^{2i}, \end{aligned} \tag{4.25}$$

where J_i^j is the complex structure defined in (4.7). We evaluate the κ-variations in the boundary limit $Y \to 0$. For simplicity, we consider only the case of constant $\dot{Y}^i/|\dot{Y}|$; that is, we take the worldsheet to be located at a fixed point on S^5.

As we have remarked, in the limit $Y \to 0$, the vielbein components $P_{\sigma}^{\underline{\mu}} = J_\sigma^i E_i^{\underline{\mu}}$ and $E_{\tau}^{\underline{Y}}$ decouple. The restriction to the AdS_5 directions entitles us to replace $(\Gamma^\mu, \Gamma^Y) \to (\gamma^\mu, \gamma^5)$, as explained in appendix A.1. Expanding the first equation in (4.23) subject to these assumptions gives

$$\begin{aligned} \delta_\kappa Z^{\mathbf{M}} E^1_{\mathbf{M}} &= 2(E_\tau^{\underline{\mu}}\gamma_{\underline{\mu}} - P_\tau^{\underline{Y}}\gamma_5)\kappa^1 \\ \delta_\kappa Z^{\mathbf{M}} E^2_{\mathbf{M}} &= 2(E_\tau^{\underline{\mu}}\gamma_{\underline{\mu}} + P_\tau^{\underline{Y}}\gamma_5)\kappa^2 . \end{aligned} \tag{4.26}$$

Let us define $\kappa \equiv \frac{1}{\sqrt{2}}(\kappa^1 + i\kappa^2)$, $\widetilde{\kappa} \equiv \frac{1}{\sqrt{2}}(\kappa^1 - i\kappa^2)$, $\kappa_\pm \equiv \frac{1\pm\gamma^5}{2}\kappa$, and $\widetilde{\kappa}_\pm \equiv \frac{1\pm\gamma^5}{2}\widetilde{\kappa}$. Then

$$\begin{aligned} \delta Z^{\mathbf{M}}(E_Q^a)_{\mathbf{M}} &= 2\left(E_\tau^{\underline{\mu}}(\gamma_{\underline{\mu}}\kappa_+)^a + P_\tau^{\underline{Y}}\widetilde{\kappa}_-^a\right), \\ \delta Z^{\mathbf{M}}(E_S^a)_{\mathbf{M}} &= 2\left(E_\tau^{\underline{\mu}}(\gamma_{\underline{\mu}}\kappa_-)^a - P_\tau^{\underline{Y}}\widetilde{\kappa}_+^a\right). \end{aligned} \tag{4.27}$$

Here and in many subsequent formulas, the SO(1, 4) spinor index has been suppressed for readability.

It will be useful for what follows to work out the properties of the various κ's under complex conjugation. The κ^I are Majorana spinors in ten dimensions; therefore, as discussed in Appendix A.2, $(\kappa^I)^* = (B \otimes B')\kappa^I$. It follows that $\kappa^* = (B \otimes B')\widetilde{\kappa}$; also,

$$\widetilde{\kappa}_- = \frac{1-\gamma^5}{2}(B^\dagger \otimes B'^\dagger)\kappa^* = (B^\dagger \otimes B'^\dagger)\frac{1+\gamma^5}{2}\kappa^* = (B^\dagger \otimes B'^\dagger)(\kappa_+)^*, \tag{4.28}$$

where the second equality is true because γ^5 and B anticommute, and the last step follows because γ^5 is real in our chosen representation. Similarly, $(\widetilde{\kappa})_+ = (B^\dagger \otimes B'^\dagger)(\kappa_-)^*$. The variations then become

$$\delta_\kappa Z^{\mathbf{M}}(E_Q^a)_{\mathbf{M}} = 2\left(E_\tau^{\underline{\mu}}(\gamma_{\underline{\mu}}\kappa_+)^a + P_\tau^{\underline{Y}}((B^\dagger \otimes B'^\dagger)(\kappa_+)^*)^a\right), \tag{4.29}$$

$$\delta_\kappa Z^{\mathbf{M}}(E_S^a)_{\mathbf{M}} = 2\left(E_\tau^{\underline{\mu}}(\gamma_{\underline{\mu}}\kappa_-)^a - P_\tau^{\underline{Y}}((B^\dagger \otimes B'^\dagger)(\kappa_-)^*)^a\right). \tag{4.30}$$

From (2.38) and (4.23) it follows that

$$\delta_\kappa Z^{\mathbf{M}}(E_Q^a)_{\mathbf{M}} \sim \delta_\kappa Z^{\mathbf{M}}(e_Q^a)_{\mathbf{M}} = Y^{-1/2}\delta_\kappa\theta^b u(\phi)_b{}^a\,, \tag{4.31}$$

where $\sim$ reminds us that we neglect terms which are subleading in the boundary limit. With this form it becomes clear that it is κ_+ which acts on the coordinate θ.

On the other hand, by substituting the bosonic part of the boundary vielbein (2.38) into (4.29), we arrive at

$$\delta Z^{\mathbf{M}}(E_Q^a)_{\mathbf{M}} \sim \frac{2}{Y}\left[\left(\dot{x}^{\underline{\mu}} + \frac{1}{2}\dot{\bar{\theta}}\gamma^{\underline{\mu}}\theta - \frac{1}{2}\bar{\theta}\gamma^{\underline{\mu}}\dot{\theta}\right)(\gamma_{\underline{\mu}}\kappa_+)^a + P_\tau^Y\left((B^\dagger\otimes B'^\dagger)\kappa_+^*\right)^a\right]. \tag{4.32}$$

Combining (4.31), (4.32) and the boundary conditions

$$\theta|_{\sigma=0} = \lambda, \qquad P_\tau^Y|_{\sigma=0} = \dot{y} \tag{4.33}$$

yields the result

$$\delta\lambda^a = \left(\dot{x}^{\underline{\mu}} + \frac{1}{2}\dot{\bar{\lambda}}\gamma^{\underline{\mu}}\lambda - \frac{1}{2}\bar{\lambda}\gamma^{\underline{\mu}}\dot{\lambda}\right)(\gamma_{\underline{\mu}}\kappa_{SYM})^a + \dot{y}\left((B^\dagger\otimes B'^\dagger)\kappa_{SYM}^*\right)^a\,, \tag{4.34}$$

with

$$\kappa_{SYM}^a = \frac{2}{Y^{1/2}}(\kappa_+)^b(u^{-1})_b{}^a\,. \tag{4.35}$$

By appendices A.1 and A.2, (4.34) is precisely the κ-variation (3.10) of the fermionic gauge theory coordinates, written in dimensionally reduced form.[4] The proper variations of the bosonic coordinates in (3.10) follow from the second equation in (4.23) and the bosonic boundary vielbein (2.38). In particular, with $E^{\underline{Y}}$ from (2.38) we find

$$\delta_\kappa Y = 0\,, \tag{4.36}$$

which says that a κ-variation does not remove the endpoints of the string worldsheet from the boundary of the AdS space.

To conclude, we have succeeded in deriving the κ-symmetry of the Wilson loop as the restriction of the stringy κ-symmetry to the boundary of AdS_5.

5. Discussion

In this paper, we studied how holography works in theories formulated in superspace. We then applied our results to the computation of Wilson loop expectation values in $\mathcal{N}=4$ super-Yang-Mills theory in four dimensions. We found that the expectation value of a loop is κ-invariant, provided the loop is smooth and lightlike, and we identified this invariance with the κ-invariance of the string worldsheet action.

[4]Strictly speaking, (4.34) gives only the transformation of the chiral component of the Majorana spinor λ. The transformation of the anti-chiral component follows from the Majorana condition.

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Intriguingly, the field theory computation shows that κ-symmetry is broken at intersections of the loop, as we saw in (3.37). It would be interesting to derive the same result from the point of view of the string worldsheet in $AdS_5 \times S^5$. The breakdown of κ-invariance in the loop may be related to the failure of the proof of (4.11) at intersections.

The structure of equation (3.37) is similar to that of the loop equation of Makeenko and Migdal [17]. Classically, $\delta_\kappa W = 0$ is equivalent to the super-Yang-Mills equations of motion, so we expect that (3.37) carries as much information as the loop equation. Since κ-variations have a well-defined geometric meaning in loop space, $\delta_\kappa W$ does not suffer from the subtlety that arises in defining the loop differential operator.

The relation we have studied between bulk and boundary superspaces seems closely connected to the relation between gauged supergravity in AdS and superconformal supergravity on the boundary [18]. It would be interesting to understand this connection better.

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A. Dirac matrices and spinors in ten dimensions

A.1 Dirac matrices

In this subsection, we explain the two decompositions of the SO(1, 9) Dirac matrices that are used in the body of the paper.

We may write the general 32×32 Dirac matrix in ten dimensions in the chiral basis

$$\hat{\Gamma}^{\hat{\mu}} = \begin{pmatrix} 0 & \Gamma^{\hat{\mu}} \\ \Gamma^{\hat{\mu}} & 0 \end{pmatrix} = \Gamma^{\hat{\mu}} \otimes \sigma^1 \,, \tag{A.1}$$

where $\Gamma^{\hat{\mu}}$ is a 16×16 block and σ^1 is a Pauli matrix. This form is appropriate when we are dealing with spinors in ten dimensions of definite SO(1, 9) chirality, such as the spinor introduced in section 3 to parametrize worldline κ-symmetry of the Wilson loop. Accordingly, the Γ-matrices used in that section are the 16×16 blocks in (A.1). Further properties of these matrices are listed in appendix A.3.

Studying spinors in $AdS_5 \times S^5$ necessitates a different and more refined decomposition, which accommodates the breaking of Lorentz symmetry from SO(1,9) to SO(1,4) × SO(5):

$$\begin{aligned}\hat{\Gamma}^{\mu} &= \gamma^{\mu} \otimes \mathbf{1} \otimes \sigma^1 \\ \hat{\Gamma}^{Y} &= \gamma^{5} \otimes \mathbf{1} \otimes \sigma^1 \\ \hat{\Gamma}^{m'} &= \mathbf{1} \otimes \gamma'^{m'} \otimes \sigma^2 \,. \end{aligned} \tag{A.2}$$

Here the γ^μ and γ^5 are the 4×4 Dirac matrices of SO(1,4) (chosen so that γ^5 is real), the $\gamma'^{m'}$ are 4×4 Dirac matrices of SO(5), $\mathbf{1}$ is the identity matrix in four dimensions, and σ^1 and σ^2 are Pauli matrices. These are the matrices that appear in the Metsaev-Tseytlin formulation of κ-symmetry reviewed in section 4.2.

A.2 Spinors

Typically, in constructing type-II Green-Schwarz superstring theory, we take our fermionic coordinates to be two Majorana-Weyl spinors Θ^I ($I = 1,2$) of SO(1,9). The information contained in these spinors can be repackaged in a single chiral (but no longer Majorana) spinor

$$\Theta = \frac{1}{\sqrt{2}}(\Theta^1 + i\Theta^2) \,. \tag{A.3}$$

The spinor Θ decomposes under SO(1,9) → SO(1,4) × SO(5) as

$$\Theta^{\hat{\alpha}} \to \Theta^a_\alpha \,, \tag{A.4}$$

where $\hat{\alpha} = 1, \ldots, 16$ is a (complex-valued) spinor index of SO(1,9), and $\alpha = 1, \ldots, 4$ and $a = 1, \ldots 4$ are spinor indices of SO(1,4) and SO(5), respectively. The conjugate spinor is defined by

$$\bar{\Theta}^\alpha_a = i((\Theta^a)^\dagger \gamma_0)^\alpha \,. \tag{A.5}$$

It is often useful to introduce a notion of chirality with respect to an SO(1,3) subgroup of the SO(1,4). From the standpoint of SO(1,4), i.e., of physics in AdS_5, this chirality is completely fictitious. However, the AdS/CFT correspondence distinguishes the 4 coordinates X^μ of AdS_5 parallel to the boundary, and in the space of these coordinates, SO(1,3) chirality is a natural concept, implemented by the matrix γ^5. Accordingly, we define the projected spinors

$$\begin{aligned}\theta^a_\alpha &= Y^{1/2}\left(\frac{1-\gamma^5}{2}\right)_\alpha^{\ \beta} \Theta^b_\beta (u(\phi)^{-1})^{\ a}_b \,, \\ \vartheta^a_\alpha &= Y^{-1/2}\left(\frac{1+\gamma^5}{2}\right)_\alpha^{\ \beta} \Theta^b_\beta (u(\phi)^{-1})^{\ a}_b \,. \end{aligned} \tag{A.6}$$

These are the coordinates we work with in section 2.4. The matrices $u(\phi)$ are the coset representatives of $\mathrm{SO}(6)/\mathrm{SO}(5)$. These coordinates are similar, but not identical to the Killing coordinates introduced in [16].

We conclude this discussion with some remarks on complex conjugation. It is possible to define a unitary 32×32 matrix $\mathcal{B}$ of complex conjugation, with the property that

$$(\hat{\Gamma}^{\hat{\mu}})^* = \mathcal{B}\hat{\Gamma}^{\hat{\mu}}\mathcal{B}^{-1} \,. \tag{A.7}$$

The complex conjugate of a Majorana spinor ζ in ten dimensions is then

$$\zeta^* = \mathcal{B}\zeta \,. \tag{A.8}$$

In the basis (A.2), the matrix of complex conjugation becomes

$$\mathcal{B} = B \otimes B' \otimes \sigma^3 \,, \tag{A.9}$$

where B and B' are the unitary matrices of complex conjugation in five-dimensional Minkowski and euclidean spaces, respectively, and σ^3 is a Pauli matrix. We do not need the explicit forms of the matrices B and B', although we will use the relation

$$(1+\gamma^5)B = B(1-\gamma^5) \,, \tag{A.10}$$

which follows from (A.7) with $\hat{\mu} = Y$.

The complex conjugate of a Majorana-Weyl spinor ζ^a_α is given in the basis (A.2) by

$$(\zeta^a_\alpha)^* = B^\beta_\alpha B'^{\,a}_b \zeta^b_\beta \,. \tag{A.11}$$

If we further decompose ζ according to the SO(1,3) chirality described above,

$$\begin{aligned}(\zeta_+)^a_\alpha &= \frac{1}{2}(1+\gamma^5)^\beta_\alpha \zeta^a_\beta \,,\\ (\zeta_-)^a_\alpha &= \frac{1}{2}(1-\gamma^5)^\beta_\alpha \zeta^a_\beta \,,\end{aligned} \tag{A.12}$$

then unitarity, (A.10), and the Majorana condition (A.11) imply the relation

$$\zeta_- = (B^\dagger \otimes B'^\dagger)(\zeta_+)^* \,. \tag{A.13}$$

A.3 Dirac matrix identities

In this subsection, we present a list [15] of Fierz and other identities satisfied by the 16-dimensional chiral Γ-matrices defined in appendix A.1. These identities are used ubiquitously (if unostentatiously) in deriving the various results of section 3.

$$\Gamma^{\hat{\mu}\,\hat{\alpha}\hat{\beta}} = \Gamma^{\hat{\mu}\,\hat{\beta}\hat{\alpha}}, \qquad \Gamma^{\hat{\mu}}_{\ \hat{\alpha}\hat{\beta}} = \Gamma^{\hat{\mu}}_{\ \hat{\beta}\hat{\alpha}} \qquad\qquad \text{(symmetry)} \tag{A.14}$$

$$\Gamma^{\hat\mu\,\hat\alpha\hat\beta}\Gamma^{\hat\nu}_{\hat\beta\hat\gamma} + \Gamma^{\hat\nu\,\hat\alpha\hat\beta}\Gamma^{\hat\mu}_{\hat\beta\hat\gamma} = 2g^{\hat\mu\hat\nu}\delta^{\hat\alpha}_{\hat\gamma} \qquad \text{(Clifford algebra)} \tag{A.15}$$

$$\Gamma^{\hat\mu}_{\hat\alpha\hat\beta}\Gamma_{\hat\mu\,\hat\gamma\hat\delta} + \Gamma^{\hat\mu}_{\hat\alpha\hat\gamma}\Gamma_{\hat\mu\,\hat\beta\hat\delta} + \Gamma^{\hat\mu}_{\hat\alpha\hat\delta}\Gamma_{\hat\mu\,\hat\beta\hat\gamma} = 0 \qquad \text{(Fierz identity)} \tag{A.16}$$

$$\Gamma^{\hat\mu}_{\hat\alpha\hat\beta}\Gamma^{\hat\alpha\hat\gamma}_{\hat\mu} = 10\delta^{\hat\gamma}_{\hat\beta} \tag{A.17}$$

$$\Gamma^{\hat\mu}_{\hat\alpha\hat\beta}\Gamma^{\hat\alpha\hat\beta}_{\hat\nu} = 16\delta^{\hat\mu}_{\hat\nu} \qquad \text{(trace)} \tag{A.18}$$

$$\Gamma_{\hat\mu}\Gamma^{\hat\nu}\Gamma^{\hat\mu} = -8\Gamma^{\hat\nu} \tag{A.19}$$

$$\Gamma^{\hat\mu\hat\nu\,\hat\alpha}{}_{\hat\beta} = \frac{1}{2}(\Gamma^{\hat\mu\,\hat\alpha\hat\gamma}\Gamma^{\hat\nu}_{\hat\gamma\hat\beta} - \Gamma^{\hat\nu\,\hat\alpha\hat\gamma}\Gamma^{\hat\mu}_{\hat\gamma\hat\beta}) \tag{A.20}$$

B. The $\mathrm{SU}(2,2|4)$ algebra

The supergroup $\mathrm{SU}(2,2|4)$ is generated by: conformal translations P_μ; Lorentz transformations $M_{\mu\nu}$; the dilatation generator D; SU(4) rotations U_i^j $(i,j = 1,\ldots,4)$; ordinary supersymmetries Q^a_α; and special supersymmetries S^a_α. The generators of SU(4) rotations may be written as $U_i^j = 2\widetilde{P}_{m'}(\widetilde{\Gamma}^{m'6})^j_i + \widetilde{M}_{m'n'}(\widetilde{\Gamma}^{m'n'})^j_i$, where the $\widetilde{P}_{m'}$ and $\widetilde{M}_{m'n'}$ $(m',n' = 1,\ldots,5)$ are generators of translations and rotations on S^5, and the $\widetilde{\Gamma}$'s are the 4×4 chiral blocks of the SO(6) Dirac matrices in the chiral basis. The generators are assigned weights according to their commutation relations with the dilatation operator: the P's have weight 1; the M's, the U's, and D itself have weight 0; the K's have weight -1; the Q's have weight $1/2$; and the S's have weight $-1/2$. The full structure of the algebra is

$$\begin{aligned}
[M_{mn}, M_{pq}] &= \eta_{m[p}M_{q]n} - \eta_{n[p}M_{q]m}\,, \\
[P_q, M_{mn}] &= \eta_{q[m}P_{n]}\,, & [K_q, M_{mn}] &= \eta_{q[m}K_{n]} \\
[D, P_m] &= P_m\,, & [D, K_m] &= -K_m \\
[P_m, K_n] &= 2\,(\eta_{mn}D + 2M_{mn}) \\
[M_{mn}, Q_\alpha{}^i] &= -\frac{1}{4}(\gamma_{mn}Q^i)_\alpha\,, & [M_{mn}, S_\alpha{}^i] &= -\frac{1}{4}(\gamma_{mn}S^i)_\alpha \\
[P_m, S_\alpha{}^i] &= (\gamma_m Q^i)_\alpha\,, & [K_m, Q_\alpha{}^i] &= (\gamma_m S^i)_\alpha \\
[D, Q_\alpha{}^i] &= \frac{1}{2}Q^i_\alpha\,, & [D, S_\alpha{}^i] &= -\frac{1}{2}S^i_\alpha\,, \\
\{Q_\alpha{}^i, \bar{Q}_j{}^\beta\} &= \delta_j{}^i(\gamma^m)_\alpha{}^b P_m\,, & \{S_\alpha{}^i, \bar{S}_j{}^\beta\} &= \delta_j{}^i(\gamma^m)_\alpha{}^b K_m\,, \\
\{Q_\alpha{}^i, \bar{S}_j{}^\beta\} &= \delta_j{}^i D + \delta_j{}^i(\gamma^{mn})_\alpha{}^\beta M_{mn} - 2\delta_\alpha{}^\beta U_j{}^i\,, \\
[U_i{}^j, Q^k_\alpha] &= \delta_i{}^k Q_\alpha{}^j - \frac{1}{4}\delta_i{}^j Q_\alpha{}^k\,, \\
[U_i{}^j, S^k_\alpha] &= \delta_i{}^k S_\alpha{}^j - \frac{1}{4}\delta_i{}^j S_\alpha{}^k\,, \\
[U_i{}^j, U_k{}^l] &= \delta_i{}^l U_k{}^j - \delta_k{}^j U_i{}^l\,,
\end{aligned} \tag{B.1}$$

together with relations that follow from these by complex conjugation. Here the γ_μ's are Dirac matrices of SO(1,4), and $\gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. All other commutators vanish.

The list of generators we have given here constitutes the *superconformal decomposition* of the SU$(2,2|4)$ superalgebra. This is the form of the algebra most convenient for the study of conformal superspace, though it is not as well adapted to physics in $AdS_5 \times S^5$. For example, the generators of translations in the X^μ directions of AdS_5 are not P_μ, but rather the linear combinations $\frac{1}{2}(P_\mu + K_\mu)$.

References

[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] P. Claus, J. Rahmfeld, H. Robins, J. Tannenhauser and Y. Zunger, *Isometries in anti-de Sitter and conformal superspaces*, *J. High Energy Phys.* **07** (2000) 047 [hep-th/0007099].

[3] J. Maldacena, *Wilson loops in large-N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002].

[4] S.-J. Rey and J. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity*, hep-th/9803001.

[5] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Superstring actions in $d = 3,4,6,10$ curved superspace*, *Phys. Lett.* **B 169** (1986) 191.

[6] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028].

[7] S. Forste, D. Ghoshal and S. Theisen, *Stringy corrections to the Wilson loop in $N = 4$ super Yang-Mills theory*, *J. High Energy Phys.* **08** (1999) 013 [hep-th/9903042].

[8] Y. Kinar, E. Schreiber, J. Sonnenschein and N. Weiss, *Quantum fluctuations of Wilson loops from string models* *Nucl. Phys.* **B 583** (2000) 76, [hep-th/9911123].

[9] N. Drukker, D.J. Gross and A. Tseytlin, *Green-Schwarz string in $AdS_5 \times S^5$: semiclassical partition function*, *J. High Energy Phys.* **04** (2000) 021 [hep-th/0001204].

[10] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces*, *Phys. Rev.* **D 60** (1999) 125006 [hep-th/9904191].

[11] E. Witten, *An interpretation of classical Yang-Mills theory*, *Phys. Lett.* **B 77** (1978) 394.

[12] E. Witten, *Twistor-like transform in ten-dimensions*, *Nucl. Phys.* **B 266** (1986) 245.

[13] L. Castellani et al., *G/H M-branes and AdS_{p+2} geometries*, *Nucl. Phys.* **B 527** (1998) 142 [hep-th/9803039].

[14] M. Rocek, W. Siegel, P. van Nieuwenhuizen and A.E. van de Ven, *Covariant superparticle quantization in a supermaxwell background*, *Phys. Lett.* **B 227** (1989) 87.

[15] J. Harnad and S. Shnider, *Constraints and field equations for ten-dimensional super Yang-Mills theory*, *Comm. Math. Phys.* **106** (1986) 183.

[16] R. Kallosh, J. Rahmfeld and A. Rajaraman, *Near horizon superspace*, *J. High Energy Phys.* **09** (1998) 002 [hep-th/9805217].

[17] A.A. Migdal, *Loop equations and* $1/N$ *expansion*, *Phys. Rep.* **102** (1983) 199.

[18] H. Liu and A.A. Tseytlin, $D=4$ *super Yang-Mills,* $D=5$ *gauged supergravity and* $D=4$ *conformal supergravity*, *Nucl. Phys.* **B 533** (1998) 88 [hep-th/9804083].